Power Systems Analysis

Chapter 2 Transmission line models

Outline

1. Line characteristics
2. Line models

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1. Line characteristics
    - Resistance r and conductance g
    - Series inductance l
    - Shunt capacitance c
    - Balanced 3ϕ lines
2. Line models

3ϕ line

Alternating currents in conductors line interact electromagnetically

Interactions couple voltages & currents across phases

In balanced operation, phases behave as if they are decoupled

In each phase, line is characterized by

- series impedance / meter $z := r + i\omega l$ Ω/m
- shunt admittance / meter to neutral $y := g + i\omega c$ Ω^{-1}/m

Assumption

Currents and charges sum to zero across all n conductors:

$$i_1(t) + \cdots + i_n(t) \;=\; 0 \quad \text{for all } t$$

$$q_1(t) + \cdots + q_n(t) \;=\; 0 \quad \text{for all } t$$

Line characteristics

1. Series inductance l

- total flux linkages λ_k of conductor k depends on all currents $i_{k'}$

$$\lambda_k = \left(\frac{\mu_0}{2\pi} \ln \frac{1}{r'_k} \right) i_k + \sum_{k' \neq k} \left(\frac{\mu_0}{2\pi} \ln \frac{1}{d_{kk'}} \right) i_{k'}$$

self inductance
henrys/m

mutual inductances
henrys/m

Line characteristics

1. Series inductance l

- total flux linkages λ_k of conductor k depends on all currents $i_{k'}$

$$\lambda_k = \left(\frac{\mu_0}{2\pi} \ln \frac{1}{r'_k} \right) i_k + \sum_{k' \neq k} \left(\frac{\mu_0}{2\pi} \ln \frac{1}{d_{kk'}} \right) i_{k'}$$

- in vector form: $\lambda = L\, i$

- Faraday's law: $v(t) = \frac{d}{dt}\lambda(t) = L \frac{d}{dt} i(t)$

  voltage drop along conductor

Line characteristics

2. Shunt capacitance c

- voltage on surface of conductor k relative to reference:

$$v_k = \left(\frac{1}{2\pi\epsilon}\ln\frac{1}{r_k}\right)q_k + \sum_{k'\neq k}\left(\frac{1}{2\pi\epsilon}\ln\frac{1}{d_{kk'}}\right)q_{k'}$$

- in vector form: $v \;=\; F\,q$

- let $C := F^{-1}$. C_{kk} : self capacitance/m, $C_{kk'}$: mutual capacitance/m

Line characteristics

2. Shunt capacitance c

- voltage on surface of conductor k relative to reference:

$$v_k = \left(\frac{1}{2\pi\epsilon} \ln \frac{1}{r_k} \right) q_k + \sum_{k' \neq k} \left(\frac{1}{2\pi\epsilon} \ln \frac{1}{d_{kk'}} \right) q_{k'}$$

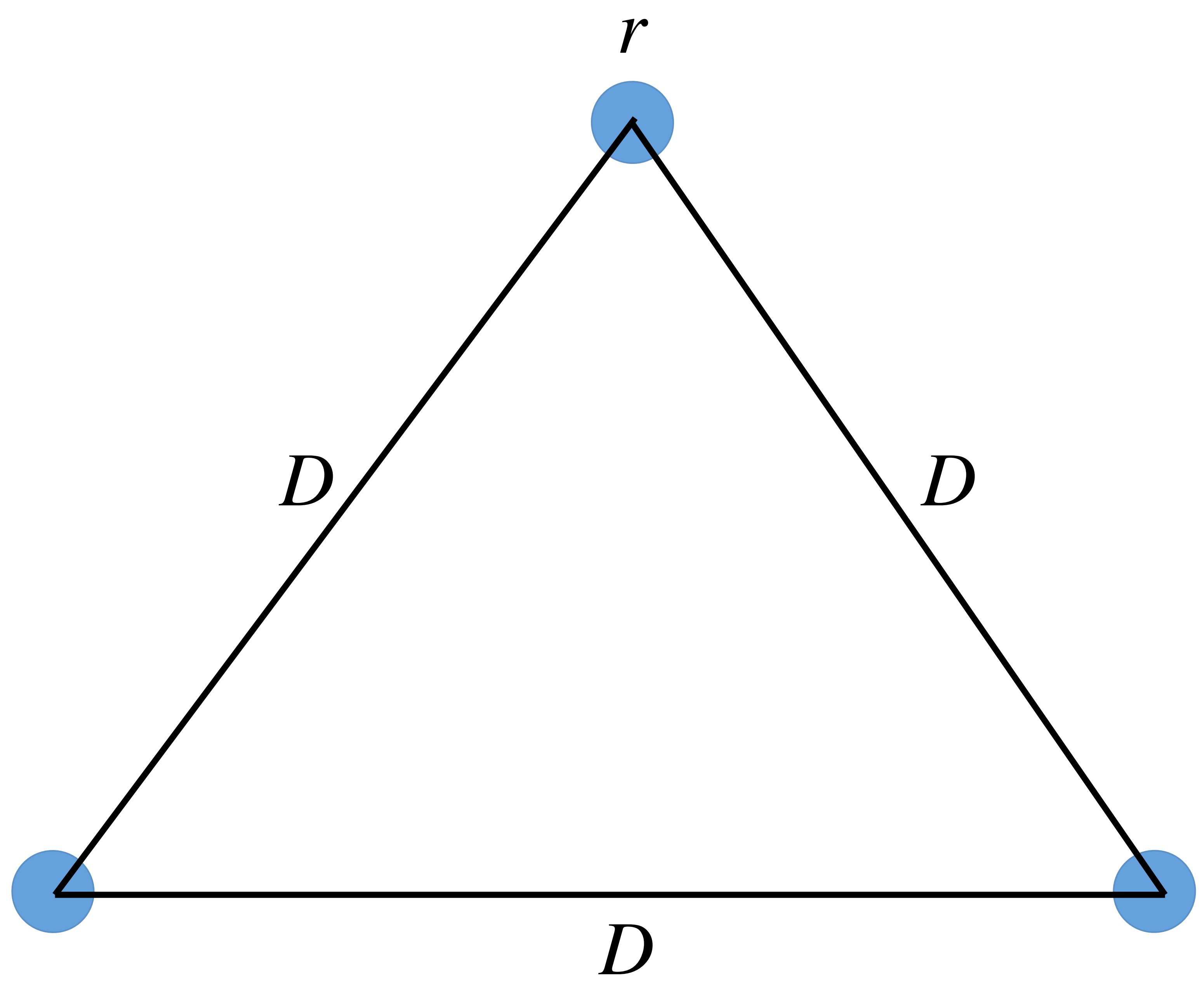
- in vector form: $v = F q$

- therefore: $\frac{d}{dt} v(t) = F\, i(t)$

Balanced 3ϕ line

Assumptions:

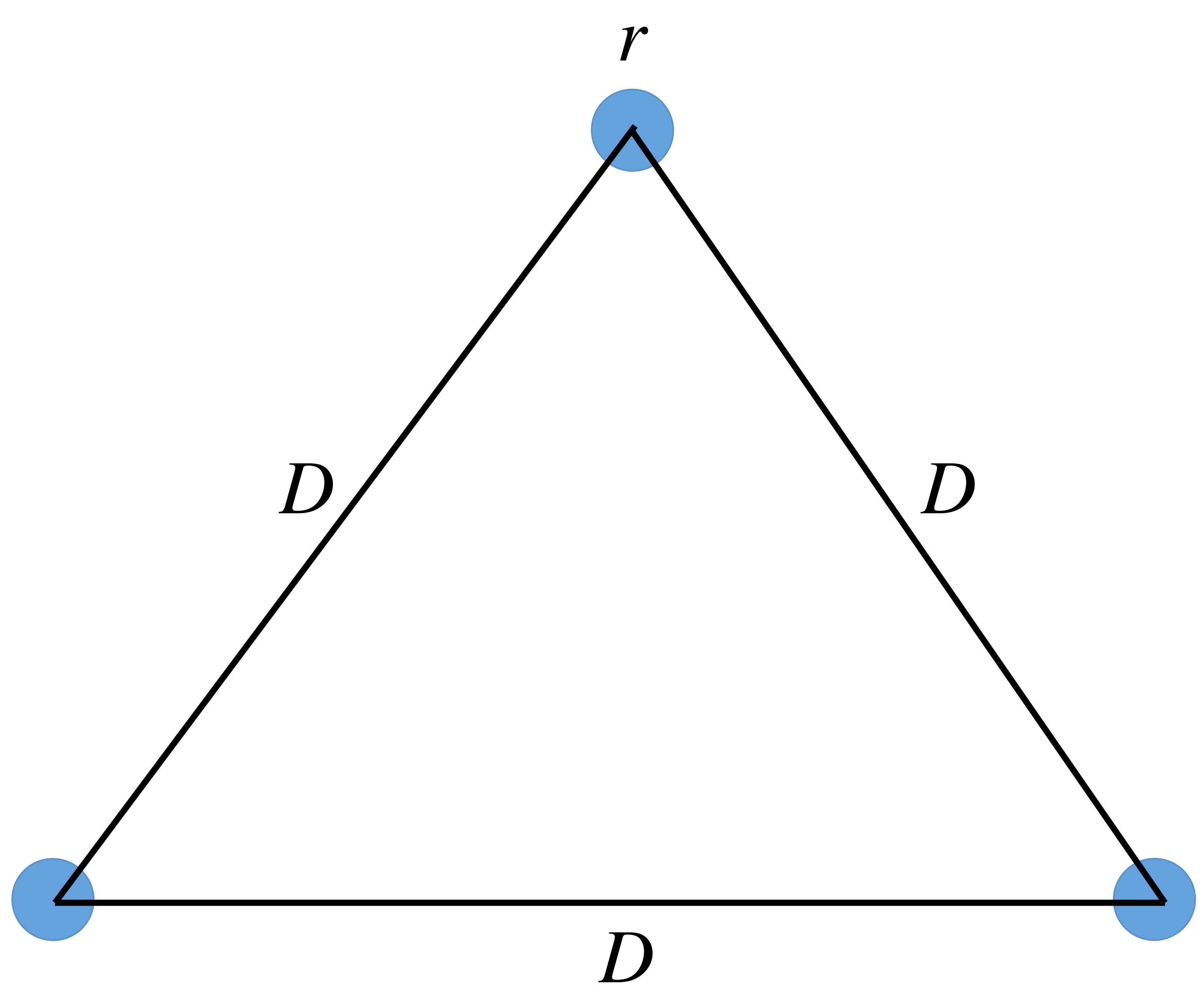
1. Conductors equally spaced at D with equal radii r
2. $i_1(t) + \cdots + i_n(t) \;=\; 0$ for all t
3. $q_1(t) + \cdots + q_n(t) \;=\; 0$ for all t



Balanced 3ϕ line

Phases are decoupled

$$\lambda_k = \underbrace{\left(\frac{\mu_0}{2\pi} \ln \frac{D}{r'} \right)}_{\text{inductance } l \text{ (H/m)}} i_k$$

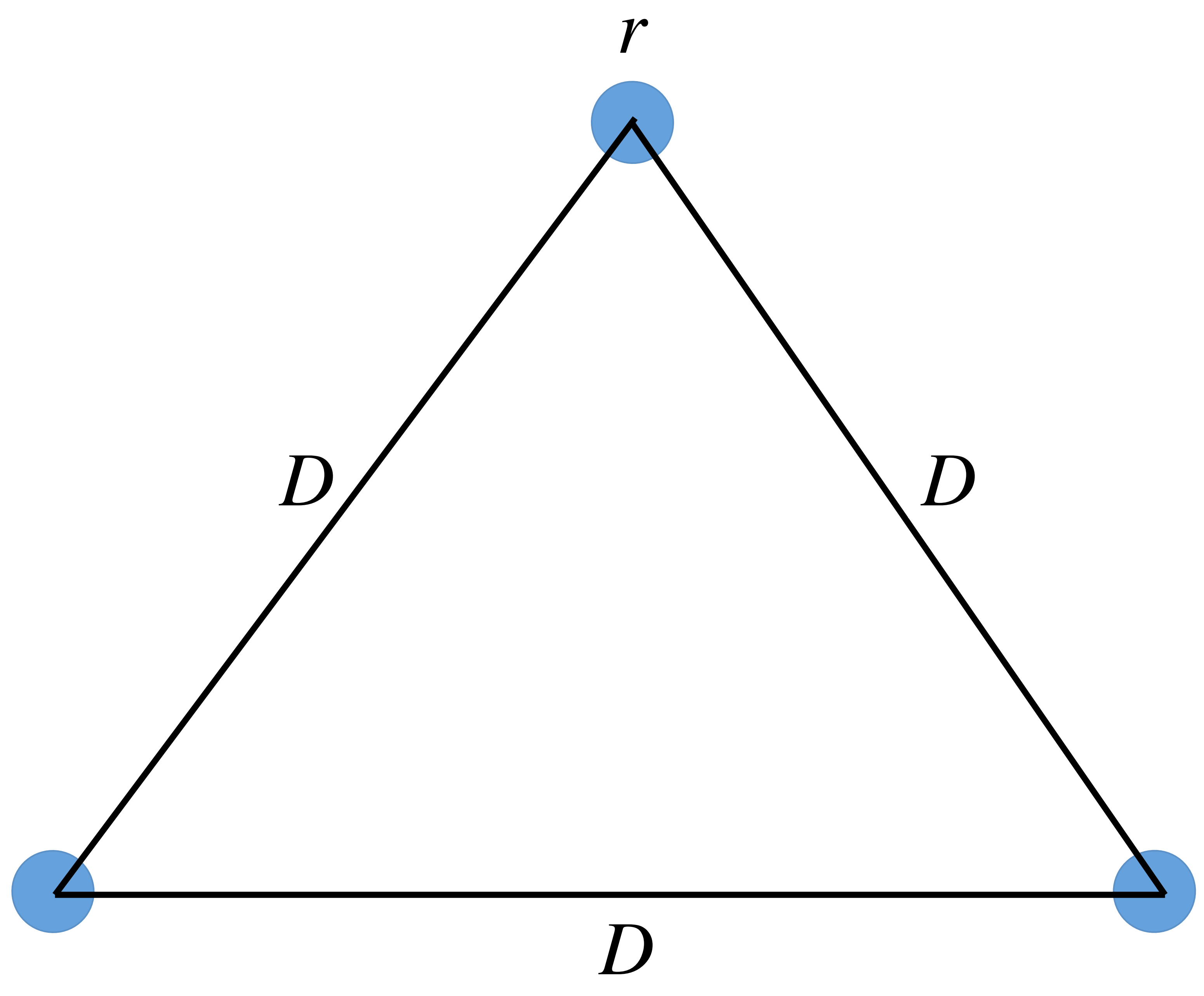


Balanced 3ϕ line

Phases are decoupled

$$\lambda_k = \underbrace{\left(\frac{\mu_0}{2\pi} \ln \frac{D}{r'} \right)}_{\text{inductance } l \text{ (H/m)}} i_k$$

$$v_k = \underbrace{\left(\frac{1}{2\pi\epsilon} \ln \frac{D}{r} \right)}_{(\,\cdot\,)^{-1}\text{: capacitance } c \text{ (F/m)}} q_k$$



Outline

1. Line characteristics
2. Line models
    - Transmission matrix
    - Π circuit model
    - Real and reactive line losses
    - Lossless line
    - Short line

Balanced 3ϕ line

Assumptions:

1. Conductors equally spaced at D with equal radii r
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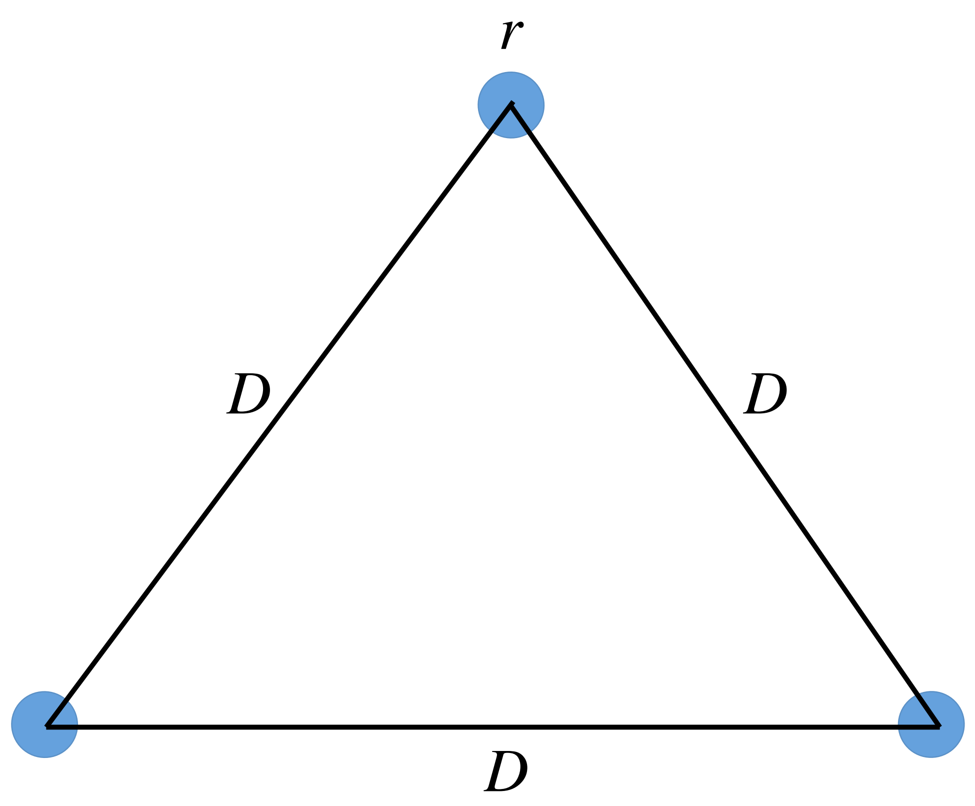
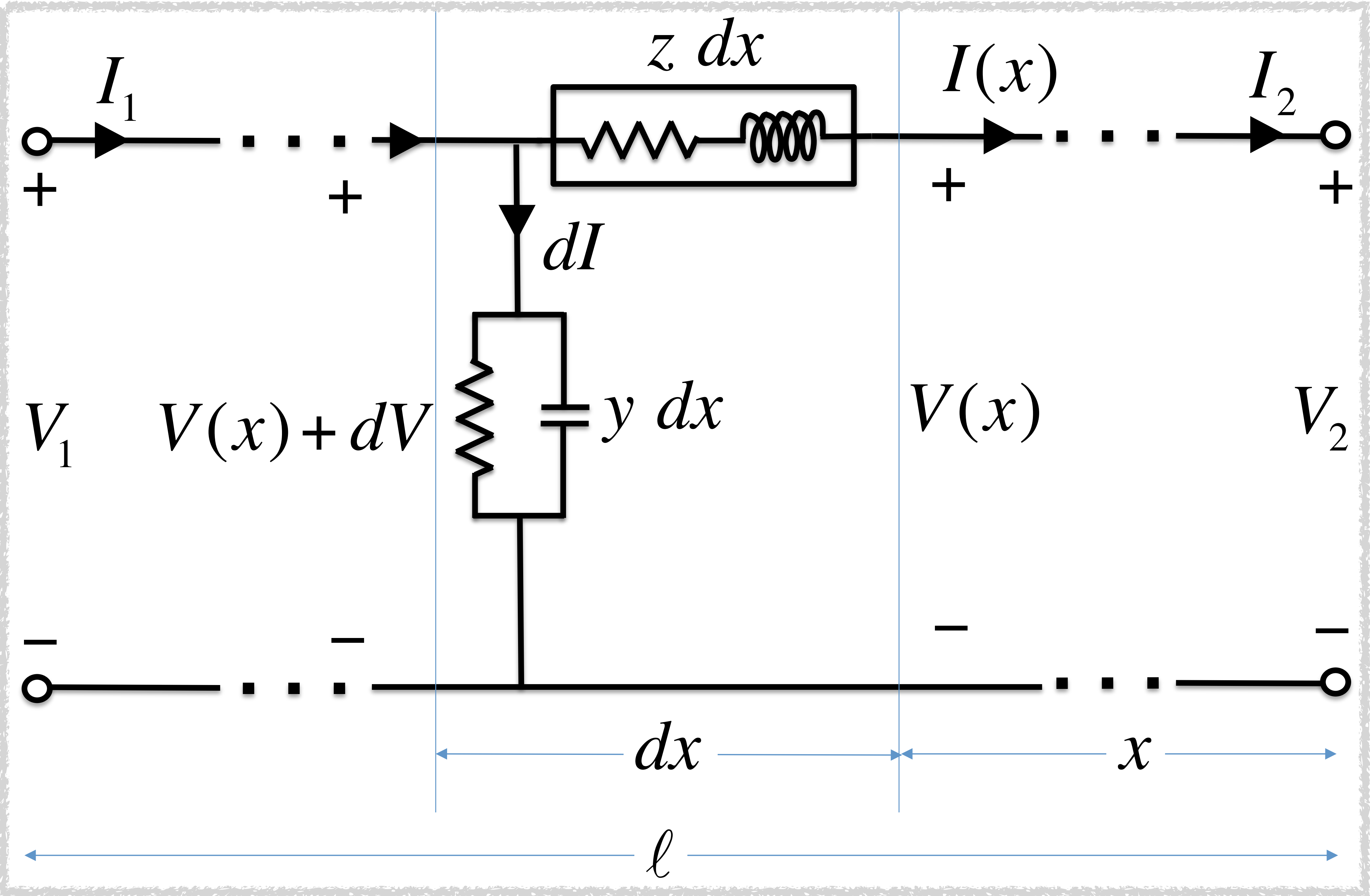


Per-phase line characteristics:

series impedance / meter $\quad z \;:=\; r + i\omega l \qquad \Omega/\text{m}$

shunt admittance / meter to neutral $\quad y \;:=\; g + i\omega c \qquad \Omega^{-1}/\text{m}$

Transmission matrix



$$dV = zI(x)\ dx$$
$$dI = (V(x) + dV)y\ dx \approx yV(x)\ dx$$

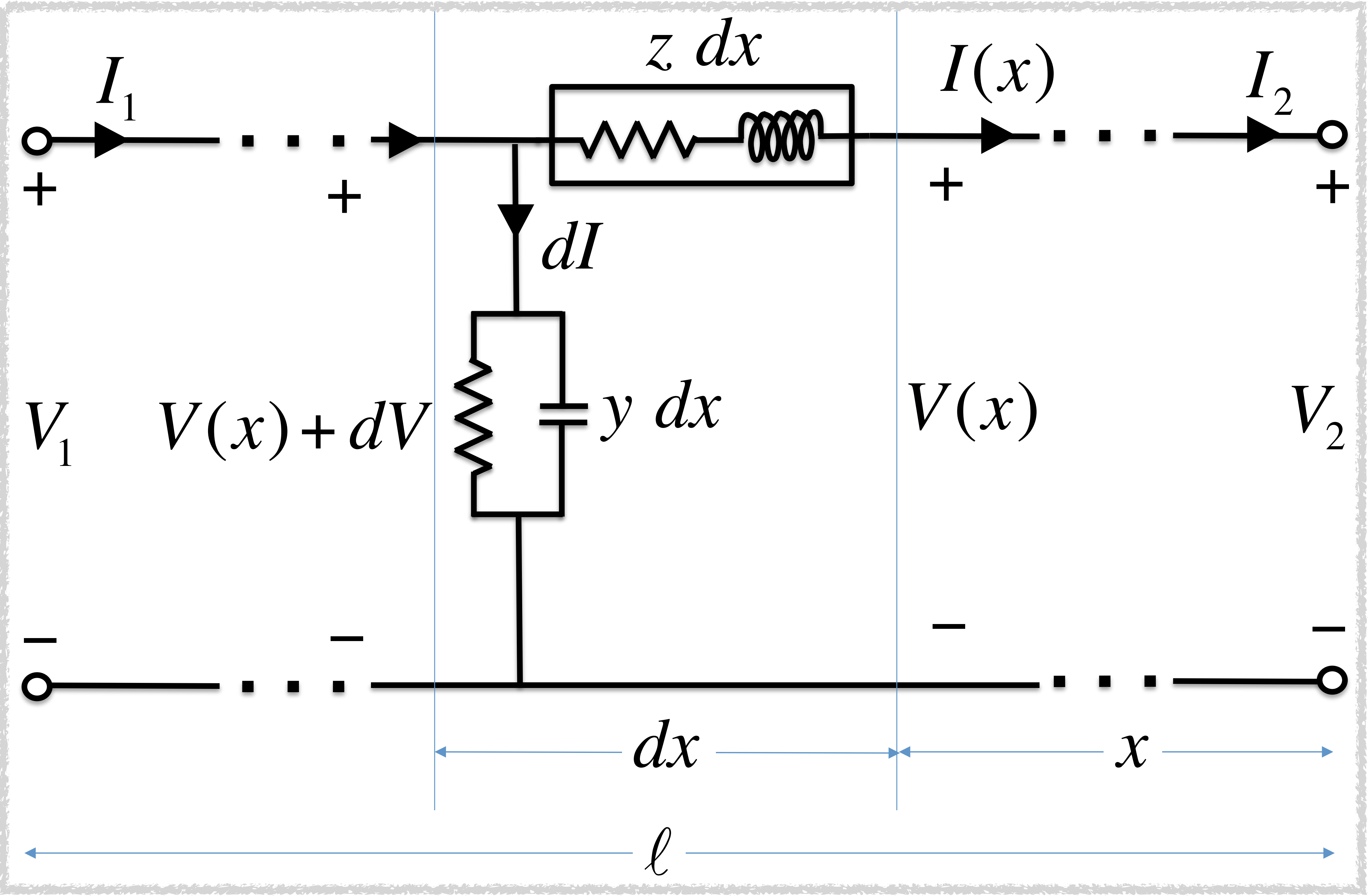
ODE:

$$\begin{bmatrix} \frac{dV}{dx} \\ \frac{dI}{dx} \end{bmatrix} = \begin{bmatrix} 0 & z \\ y & 0 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix}$$

boundary cond:

$$V(0) = V_2,\ I(0) = I_2$$

Transmission matrix



$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = U \begin{bmatrix} e^{\gamma x} & 0 \\ 0 & e^{-\gamma x} \end{bmatrix} U^{-1} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

$$U := \begin{bmatrix} Z_c & -Z_c \\ 1 & 1 \end{bmatrix}, \quad U^{-1} := \frac{1}{2Z_c} \begin{bmatrix} 1 & Z_c \\ -1 & Z_c \end{bmatrix}$$

characteristic impedance $Z_c := \sqrt{\frac{z}{y}} \quad \Omega/m$

propagation constant $\gamma := \sqrt{zy} \quad m^{-1}$

Transmission matrix

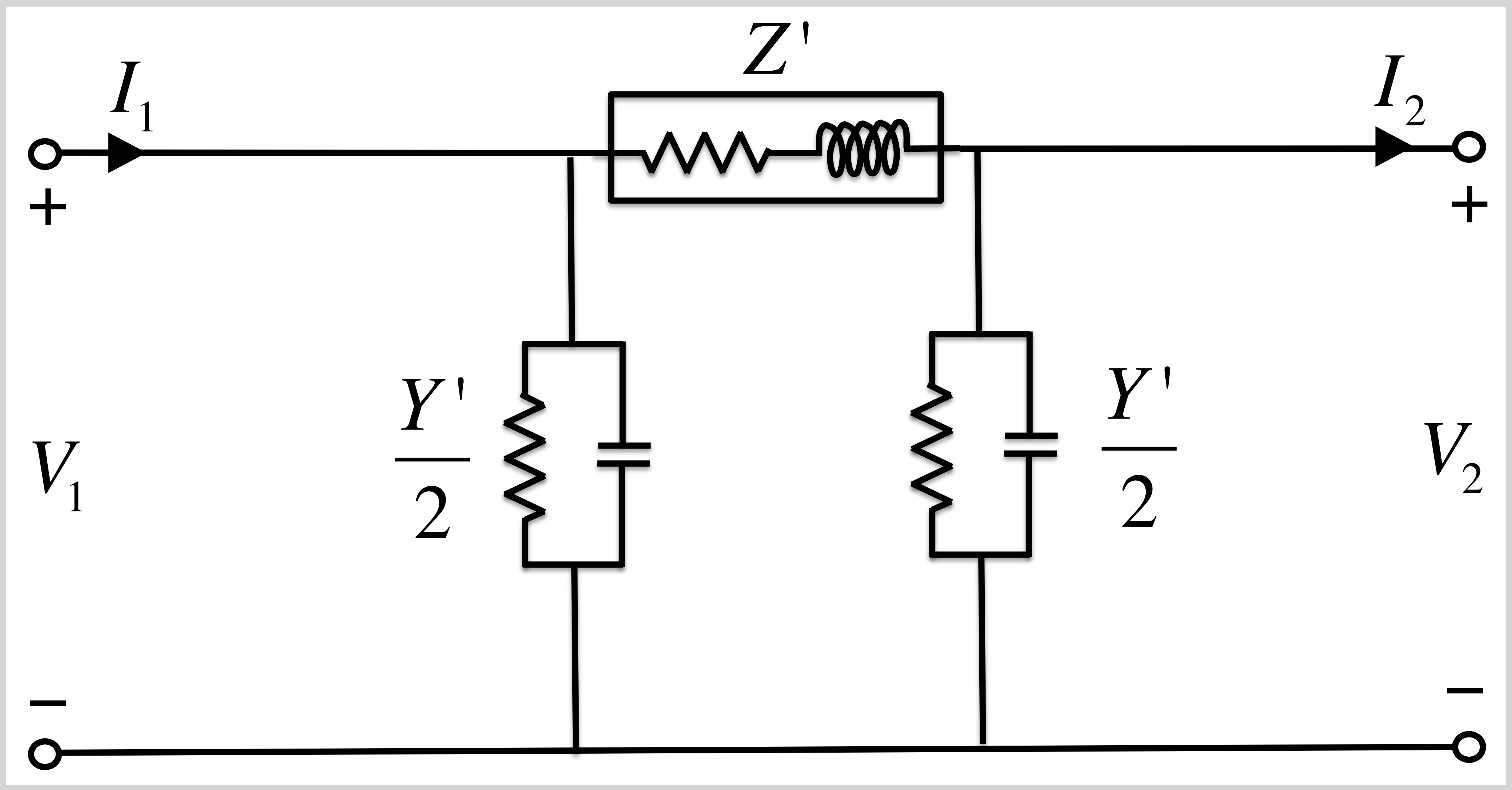
Transmission matrix maps receiving-end (V_2, I_2) to sending-end (V_1, I_1)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh(\gamma\ell) & Z_c \sinh(\gamma\ell) \\ Z_c^{-1} \sinh(\gamma\ell) & \cosh(\gamma\ell) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

characteristic impedance $Z_c \ := \ \sqrt{\frac{z}{y}} \quad \Omega/m$

propagation constant $\gamma \ := \sqrt{zy} \quad m^{-1}$

Π circuit model



Kirchhoff's laws:

$$I_1 = \frac{Y'}{2}V_1 + \frac{Y'}{2}V_2 + I_2$$

$$V_1 - V_2 = Z'\left(\frac{Y'}{2}V_2 + I_2\right)$$

$$Z' = Z_c \sinh(\gamma\ell) = \sqrt{\frac{z}{y}}\sinh(\gamma\ell) = Z\,\frac{\sinh(\gamma\ell)}{\gamma\ell}$$

$$\frac{Y'}{2} = \frac{1}{Z_c}\,\frac{\cosh(\gamma\ell)-1}{\sinh(\gamma\ell)} = \frac{1}{Z_c}\,\frac{\sinh(\gamma\ell/2)}{\cosh(\gamma\ell/2)} = \frac{Y}{2}\,\frac{\tanh(\gamma\ell/2)}{\gamma\ell/2}$$

$Z := z\ell, \; Y := y\ell$

Π circuit model

Long line ($\ell > 150$ miles) : use Z' and Y'

Medium line ($50 < \ell < 150$ miles) : use $Z = z\ell$ and $Y = i\omega C$

Short line ($\ell < 50$ miles) : use $Z = z\ell$ and $Y = 0$

Line loss

Sending-end current

$$I_{12} = \frac{1}{Z'}(V_1 - V_2) + \frac{Y'}{2}V_1$$

$$I_{21} = \frac{1}{Z'}(V_2 - V_1) + \frac{Y'}{2}V_2$$

$$\left(I_{12} = I_1,\ I_{21} = -I_2\right)$$

Real and reactive line losses

$$I_{12} + I_{21} = (Y'/2)^H \left(|V_1|^2 + |V_2|^2 \right)$$

If $Y' = 0$ then $I_{12} = -I_{21}$ sending current = receiving current

Line loss

Sending-end line power

$$S_{12} := V_1 I_{12}^H = \left(\frac{1}{Z'}\right)^H \left(|V_1|^2 - V_1 V_2^H\right) + \left(\frac{Y'}{2}\right)^H |V_1|^2$$

$$S_{21} := V_2 (I_{21})^H = \left(\frac{1}{Z'}\right)^H \left(|V_2|^2 - V_2 V_1^H\right) + \left(\frac{Y'}{2}\right)^H |V_2|^2$$

Real and reactive line losses

$$S_{12} + S_{21} = Z' |I_{12}^s|^2 + \left(\frac{Y'}{2}\right)^H \left(|V_1|^2 + |V_2|^2\right)$$

Lossless line: $r = g = 0$

Characteristic impedance is real

$$Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{i\omega l}{i\omega c}} = \sqrt{\frac{l}{c}} \quad \Omega$$

Propagation constant is imaginary

$$\gamma = \sqrt{zy} = \sqrt{(i\omega l)(i\omega c)} = i\omega\sqrt{lc} \quad m^{-1}$$

Π circuit model: both series impedance and shunt admittance are reactive:

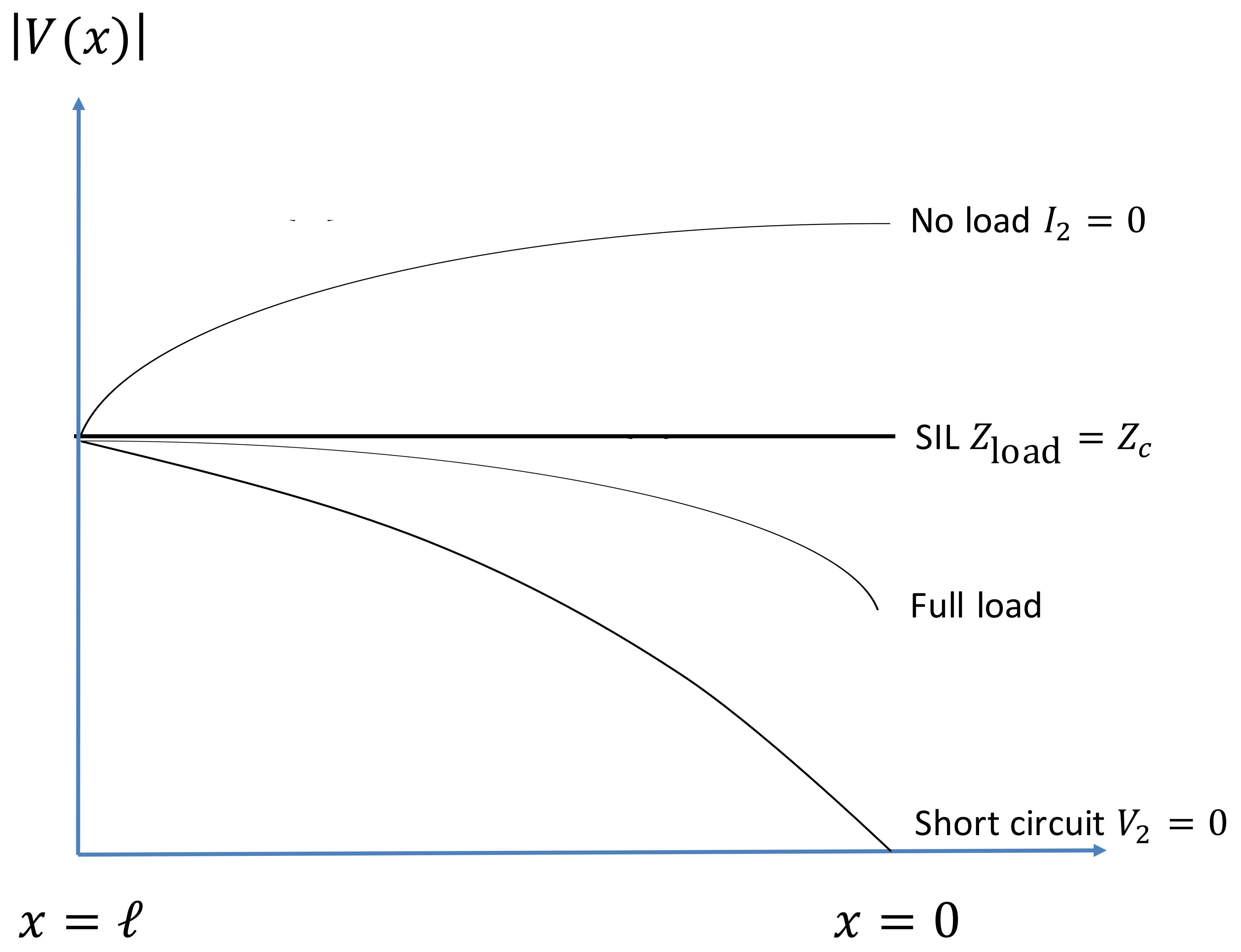
$$Z' = i\,Z_c \sin(\beta\ell) \quad \Omega, \qquad \frac{Y'}{2} = i\frac{\omega c\ell}{2}\frac{\tan(\beta\ell/2)}{\beta\ell/2} \quad \Omega^{-1}$$

Lossless line: $r = g = 0$

Voltage along the line

$$V(x) = V_2 \cos(\beta x) + i Z_c I_2 \sin(\beta x)$$

$$\beta := \omega\sqrt{lc}$$



Generally voltage drops along the line towards load

Short line: $Y = 0$

Sending-end power from i to j:

$$S_{ij} = V_i I_{ij}^* = V_i \frac{V_i^* - V_j^*}{Z^*} = \frac{1}{Z^*} \left(|V_i|^2 - V_i V_j^* \right)$$

Short line: $Y = 0$

Load voltage solution and collapse

Receiving-end load power at bus 2:

$$-S_{21} = -V_2 I_{21}^* = -\frac{1}{Z^*}\left(|V_2|^2 - V_2 V_1^* \right)$$

Load power: $P + iQ := -S_{21}$

Express $-S_{21}$ in terms of load power P to relate load voltage $|V_2|$ to P

$$-S_{21} = P(1 + i \tan\phi)$$

$\phi := \theta_{V_2} - \theta_{-I_{21}}$: load power factor angle

Short line: $Y = 0$

Load voltage solution and collapse

How does load voltage $|V_2|$ depend on active load power P ?

$$P(1 + i\tan\phi) = -\frac{1}{Z^*}\left(|V_2|^2 - |V_2||V_1|e^{i\theta_{21}}\right)$$

Assume: $V_1 := |V_1|\angle 0° \ \Rightarrow\ \theta_{21} := \theta_2 - \theta_1 = \theta_2$

- 2 real equations in $|V_2|$ and θ_{21} with P as parameter
- Solve for load voltage $|V_2|$ given any P
- As load power P increases, solutions $|V_2|$ trace out a nose curve
- If P increases further, no real solutions for $|V_2|$ exists - voltage collapse

Short & lossless line: $R = 0, \ Y = 0$

Sending-end power from i to j:

$$S_{ij} = \frac{i}{X}\left(|V_i|^2 - V_i V_j^* \right)$$

Hence

$$P_{12} = \frac{|V_1||V_2|}{X} \sin\theta_{12}$$

$$Q_{12} = \frac{1}{X}\left(|V_1|^2 - |V_1||V_2|\cos\theta_{12} \right)$$

$$Q_{21} = \frac{1}{X}\left(|V_2|^2 - |V_1||V_2|\cos\theta_{12} \right)$$

Short & lossless line: $R = 0,\ Y = 0$

1. DC power flow model: $R = 0$, fixed $|V_i|$, $\sin\theta_{12} \approx \theta_{12}$, ignore Q_{ij}

$$P_{ij} = \frac{|V_1||V_2|}{X}\,\theta_{12}$$

2. Decoupling:

$$\frac{\partial P_{12}}{\partial |V_i|} = \frac{|V_j|}{X}\sin\theta_{12} \approx 0 \qquad \frac{\partial Q_{ij}}{\partial \theta_{12}} = \frac{|V_1||V_2|}{X}\sin\theta_{12} \approx 0$$

$$\frac{\partial P_{12}}{\partial \theta_{12}} = \frac{|V_1||V_2|}{X}\cos\theta_{12} \approx \frac{|V_1||V_2|}{X}$$

Short & lossless line: $R = 0,\ Y = 0$

3. Voltage regulation

$$\frac{\partial Q_{12}}{\partial |V_2|} = -\frac{|V_1|}{X} \cos\theta_{12} \ < \ 0$$

$$\frac{\partial Q_{21}}{\partial |V_2|} = \frac{1}{X}\left(2|V_2| - |V_1|\cos\theta_{12}\right) \ > \ 0$$

Voltage regulation: maintain high $|V_2|$:

- decrease Q_{12}
- increase Q_{21}