# Power Systems Analysis

## Chapter 5 Bus injection models

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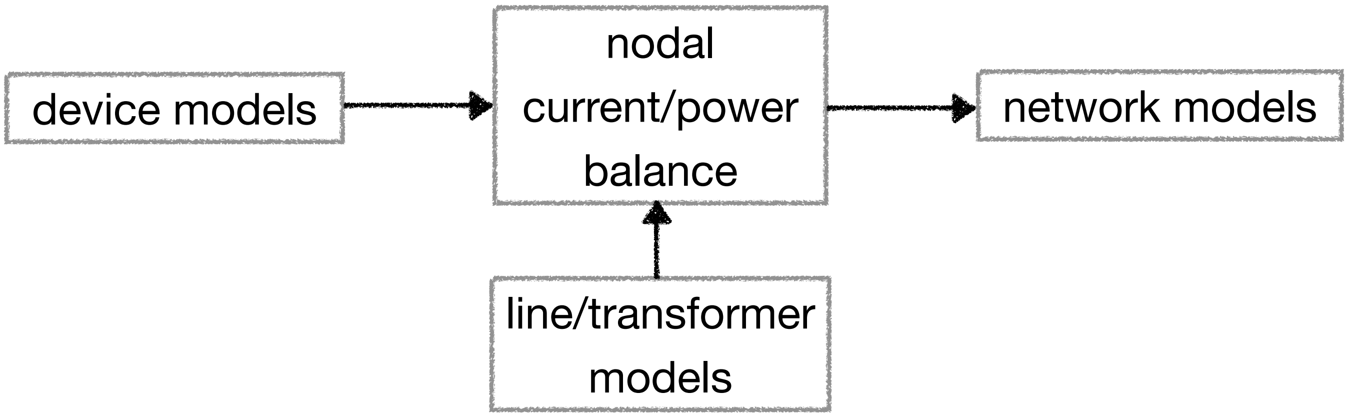
# Outline

1. Component models
2. Network model: $VI$ relation
3. Network model: $Vs$ relation
4. Computation methods

# Outline

1. Component models
    - Sources, impedance
    - Line
    - Transformer
2. Network model: $VI$ relation
3. Network model: $Vs$ relation
4. Computation methods

# Overview



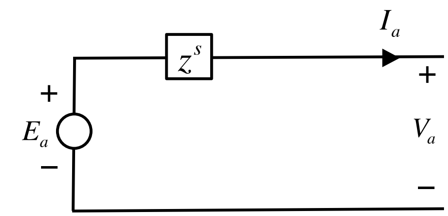

single-phase or 3-phase

# Single-phase devices

1. Single-terminal device $j$
    - Voltage source $\left(E_j, z_j\right)$, current source $\left(J_j, y_j\right)$, power source $\left(\sigma_j, z_j\right)$, impedance $z_j$
    - Terminal variables $\left(V_j, I_j, s_j\right)$
    - External model: relation between $\left(V_j, I_j\right)$ or $\left(V_j, s_j\right)$
2. Two-terminal device $(j, k)$
    - Line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$, transformer $\left(n_{jk}, y^s_{jk}, y^m_{jk}\right)$
    - Terminal variables $\left(V_j, I_{jk}, S_{jk}\right)$ and $\left(V_k, I_{kj}, S_{kj}\right)$
    - External model: relation between $\left(V_j, V_k, I_{jk}, I_{kj}\right)$ or $\left(V_j, V_k, S_{jk}, S_{kj}\right)$

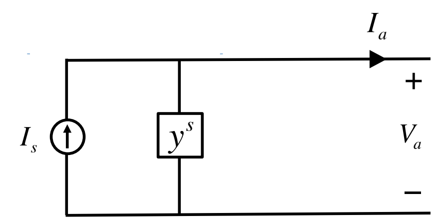
# Single-phase devices

1. Voltage source $\left(E_j, z_j\right)$
   - Constant internal voltage $E_j$ with series impedance $z_j$
   - Models for Thevenin equivalent circuit of a balanced synchronous machine, secondary side of transformer, grid-forming inverter
   - External model: $V_j = E_j - z_j I_j$
   - External model: $s_j \;=\; V_j I_j^{\mathsf{H}} \;=\; y_j^{\mathsf{H}}\, V_j \left(E_j - V_j\right)^{\mathsf{H}}$



# Single-phase devices

2. Current source $\left(J_j, y_j\right)$
   - Constant internal current $J_j$ with shunt admittance $y_j$
   - Models for Norton equivalent circuit of a synchronous generator, load (e.g. electric vehicle charger), grid-following inverter
   - External model: $I_j = J_j - y_j V_j$
   - External model: $s_j \;=\; V_j I_j^{\mathsf{H}} \;=\; V_j \left(J_j - y_j V_j\right)^{\mathsf{H}}$



# Single-phase devices

3. Power source $\left(\sigma_j, z_j\right)$

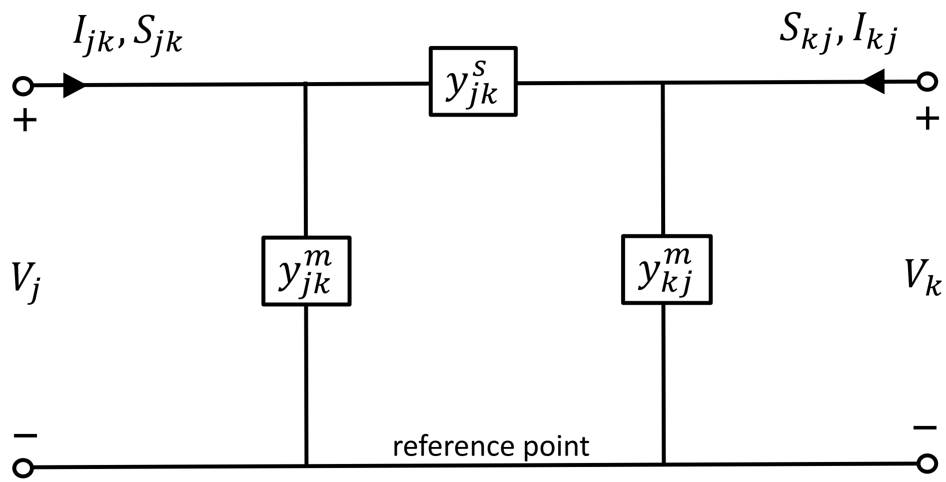
- Constant internal power $\sigma_j$ in series with impedance $z_j$
- Models for load, generator, secondary side of transformer
- External model: $\sigma_j = \left(V_j - z_j I_j\right) I_j^{\mathsf{H}}$
- External model: $s_j \ = \ V_j I_j^{\mathsf{H}} \ = \ \sigma_j + z_j I_j I_j^{\mathsf{H}}$

# Single-phase devices

4. Impedance $z_j$
   - Constant impedance $z$
   - Models for load
   - External model: $V_j = z_j I_j$
   - External model: $s_j \;=\; V_j I_j^{\mathsf{H}} \;=\; \dfrac{\left|V_j\right|^2}{z_j^{\mathsf{H}}}$

# Single-phase line $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$

## $VI$ relation: $\Pi$ circuit and admittance matrix $Y_{\text{line}}$




$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y_{jk}^s + y_{jk}^m & -y_{jk}^s \\ -y_{jk}^s & y_{jk}^s + y_{kj}^m \end{bmatrix}}_{Y_{\text{line}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

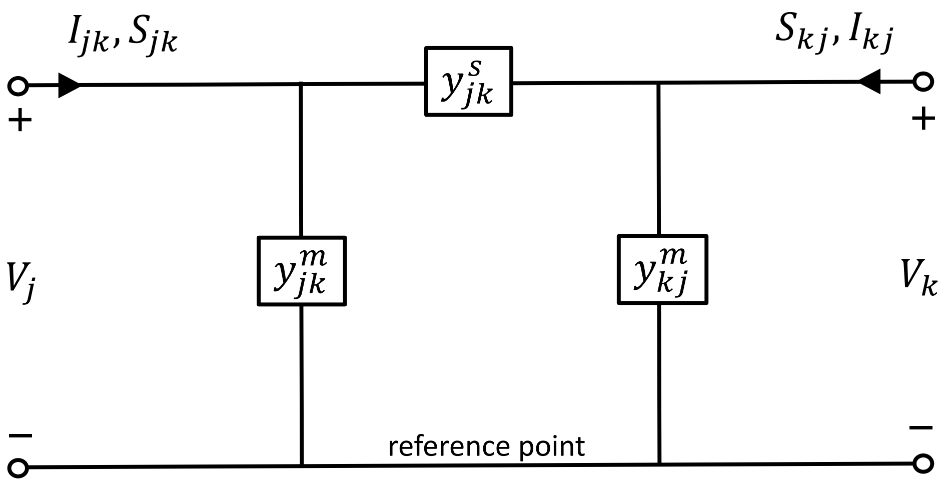
$$I_{jk} = y_{jk}^s(V_j - V_k) + y_{jk}^m V_j,$$
$$I_{kj} = y_{jk}^s(V_k - V_j) + y_{kj}^m V_k$$

admittance matrix $Y_{\text{line}}$ :

- complex symmetric
- $[Y]_{jk} = -$ series admittance

# Single-phase line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$

## $VI$ relation: $\Pi$ circuit and admittance matrix $Y_{\text{line}}$




$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j,$$
$$I_{kj} = y^s_{jk}(V_k - V_j) + y^m_{kj} V_k$$

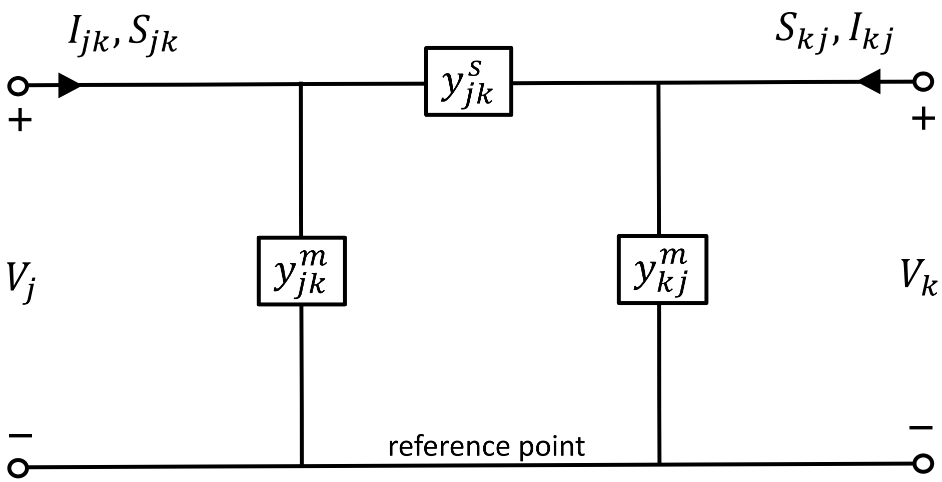
Their sum is total line current loss

$$I_{jk} + I_{kj} = y^m_{jk} V_j + y^m_{kj} V_k \neq 0$$

If $y^m_{jk} = y^m_{kj} = 0$, then $I_{jk} = - I_{kj}$

# Single-phase line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$

## $Vs$ relation



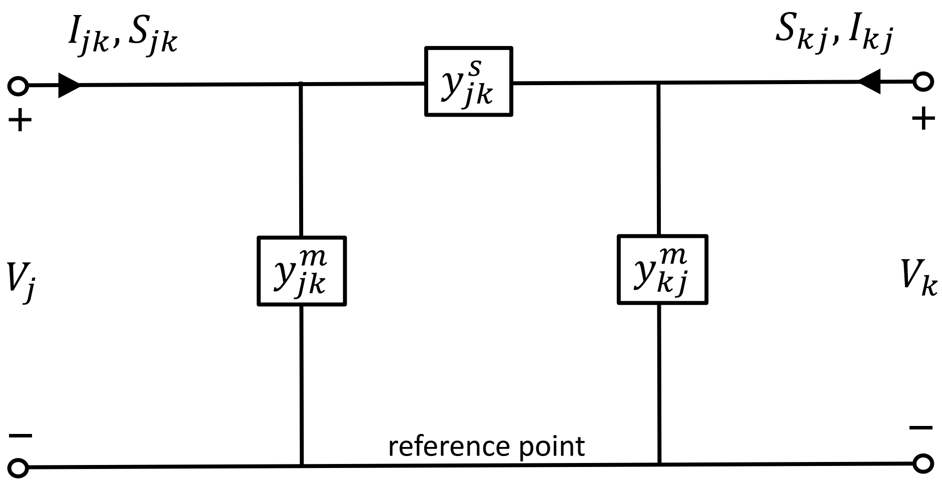

$$S_{jk} \;:=\; V_j I_{jk}^H \;=\; \left(y^s_{jk}\right)^H \left(|V_j|^2 - V_j V_k^H\right) \;+\; \left(y^m_{jk}\right)^H |V_j|^2$$

$$S_{kj} \;:=\; V_k I_{kj}^H \;=\; \left(y^s_{jk}\right)^H \left(|V_k|^2 - V_k V_j^H\right) \;+\; \left(y^m_{kj}\right)^H |V_k|^2$$

quadratic equations

# Single-phase line $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$

## $Vs$ relation



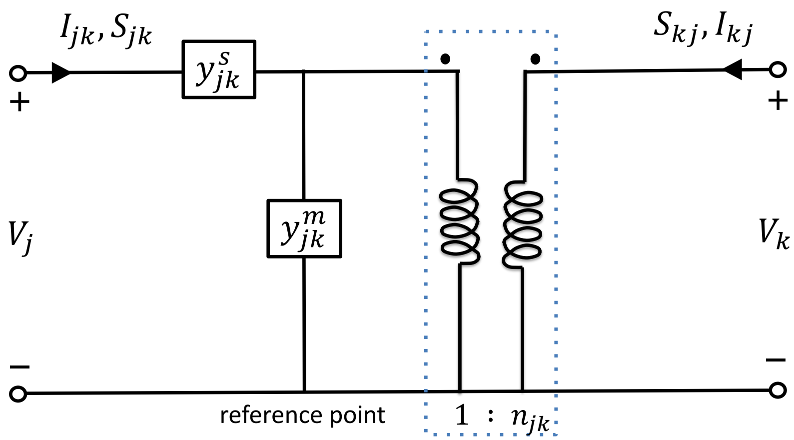

Line loss

$$S_{jk} + S_{kj} = \left(y_{jk}^s\right)^H \left|V_j - V_k\right|^2 + \left(y_{jk}^m\right)^H |V_j|^2 + \left(y_{kj}^m\right)^H |V_k|^2$$

series loss　　shunt loss

# Single-phase transformer $\left(K\left(n_{jk}\right), y^s_{jk}, y^m_{jk}\right)$

## Real $K\left(n_{jk}\right) = n_{jk}$




$$I_{jk} = y^s_{jk}\left(V_j - a_{jk}V_k\right)$$
$$I_{jk} = y^m_{jk}\, a_{jk}V_k + n_{jk}(-I_{kj})$$

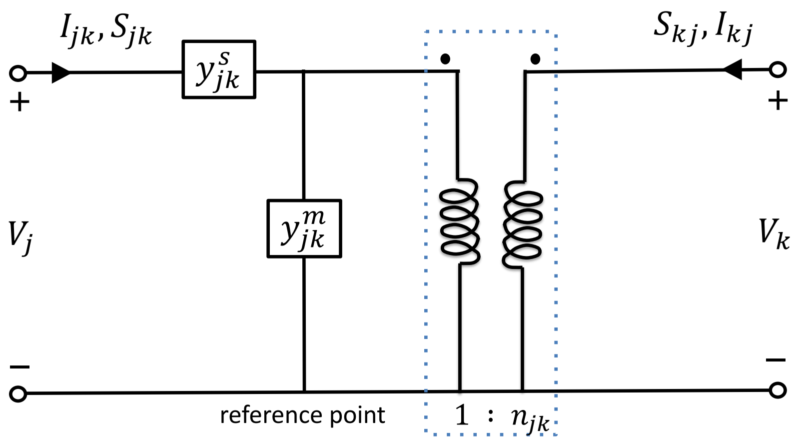
$Y_{\text{transformer}}$ : complex symmetric

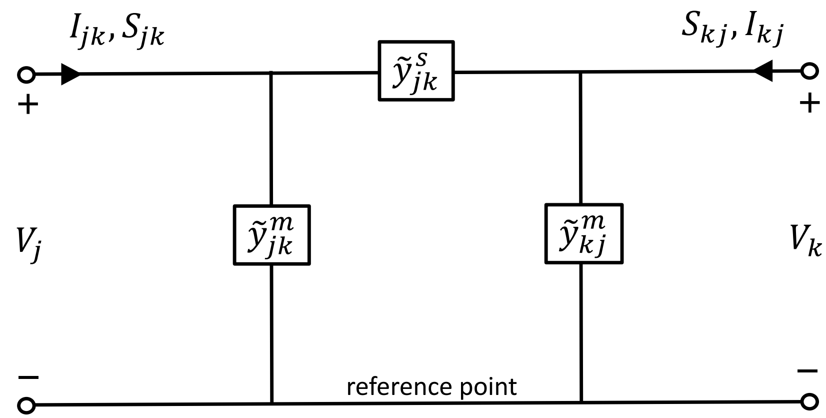
Hence: admittance matrix with equivalent $\Pi$ circuit

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y^s_{jk} & -a_{jk}\, y^s_{jk} \\ -a_{jk}\, y^s_{jk} & a^2_{jk}\left(y^s_{jk} + y^m_{jk}\right) \end{bmatrix}}_{Y_{\text{transformer}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

# Single-phase transformer $\left(K\left(n_{jk}\right), y_{jk}^s, y_{jk}^m\right)$

## Real $K\left(n_{jk}\right) = n_{jk}$




$$I_{jk} = y_{jk}^s \left(V_j - a_{jk} V_k\right)$$

$$I_{jk} = y_{jk}^m a_{jk} V_k + n_{jk}(-I_{kj})$$

$$\tilde{y}_{jk}^s := a_{jk}\, y_{jk}^s$$

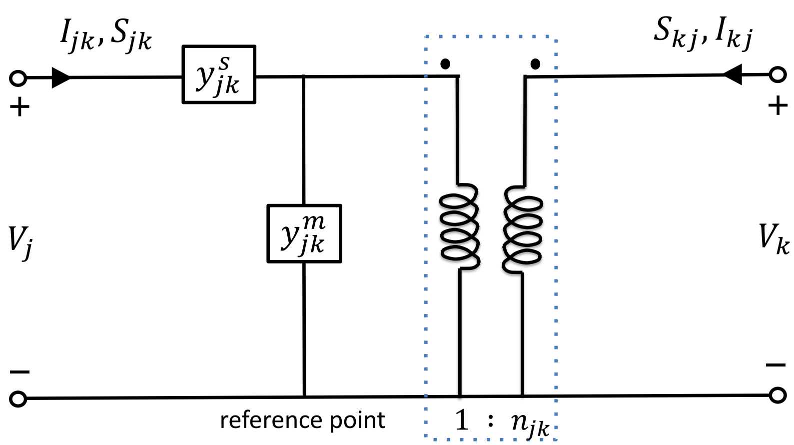
$$\tilde{y}_{jk}^m := (1 - a_{jk}) y_{jk}^s$$

$$\tilde{y}_{kj}^m := a_{jk}(a_{jk} - 1) y_{jk}^s + a_{jk}^2\, y_{jk}^m$$

$\tilde{y}_{jk}^m \neq \tilde{y}_{kj}^m$

# Single-phase transformer $\left(K\left(n_{jk}\right), y_{jk}^s, y_{jk}^m\right)$

## Complex $K\left(n_{jk}\right)$



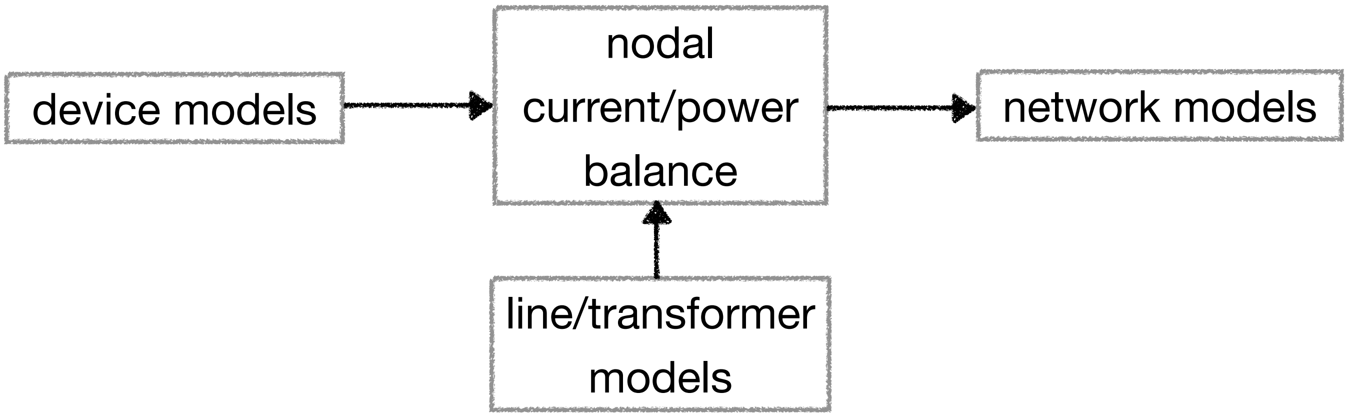

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y_{jk}^s & -y_{jk}^s / K_{jk}(n) \\ -y_{jk}^s / K_{jk}^{\mathsf{H}}(n) & \left(y_{jk}^s + y_{jk}^m\right) / |K_{jk}(n)|^2 \end{bmatrix}}_{Y_{\text{transformer}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

- $Y_{\text{transformer}}$ : *not* complex symmetric
- Has no equivalent $\Pi$ circuit
- Use transmission matrix for analysis

# Outline

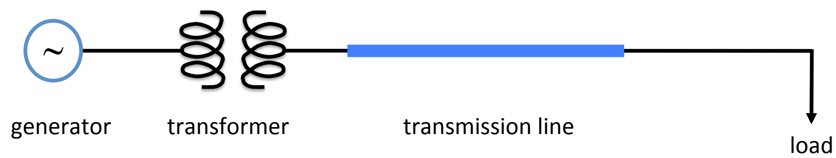
1. Component models
2. Network model: $VI$ relation
    - Examples
    - $VI$ relation (admittance matrix $Y$)
    - Kron reduction
    - Invertibility of $Y$
3. Network model: $Vs$ relation
4. Computation methods

# Network model




single-phase or 3-phase

# Example




**System**

- Generator: current source $(I_1, y_1)$
- Transformer $(n, y^l, y^m)$
- Transmission line with series admittance $y$
- Load: current source $(I_2, y_2)$

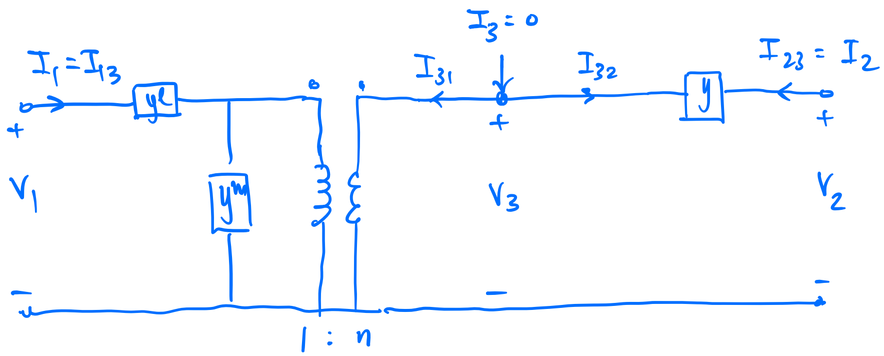
**Derive**

- Derive network model (admittance matrix $Y$)

Derive $Y$ in 2 steps

# Example

## Step 1: transformer + line




$$\begin{bmatrix} I_{13} \\ I_{31} \end{bmatrix} = \begin{bmatrix} y^l & -a\,y^l \\ -a\,y^l & a^2\left(y^l + y^m\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix}$$

relate branch currents with nodal voltages

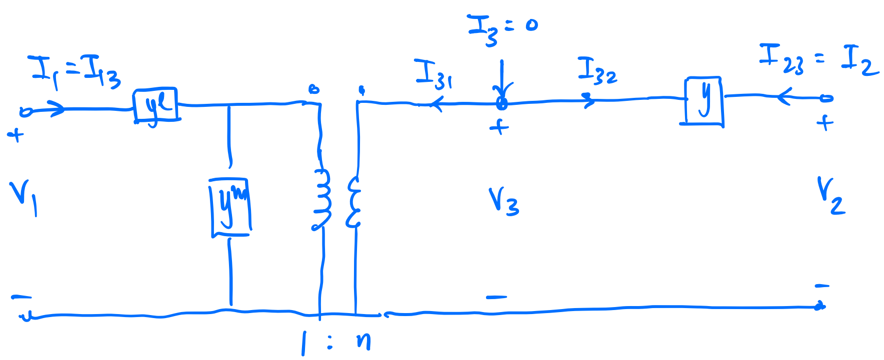
$$\begin{bmatrix} I_{32} \\ I_{23} \end{bmatrix} = \begin{bmatrix} y & -y \\ -y & y \end{bmatrix} \begin{bmatrix} V_3 \\ V_2 \end{bmatrix}$$

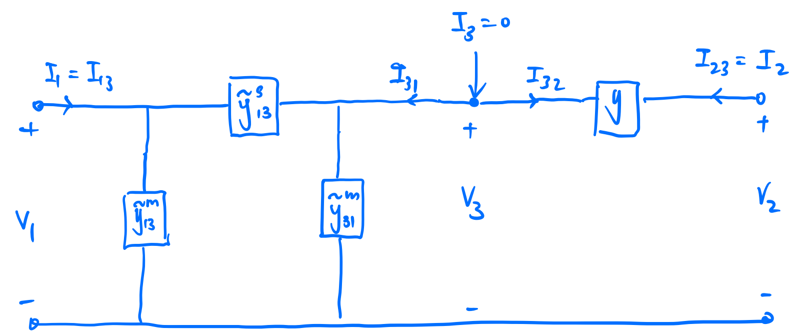
Nodal current balance (KCL):

$$I_1 = I_{13}$$
$$I_3 = I_{31} + I_{32} = 0$$
$$I_2 = I_{23}$$

# Example

## Step 1: transformer + line



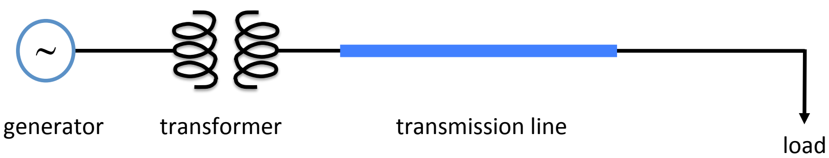

Eliminate branch currents:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} y^l & 0 & -ay^l \\ 0 & y & -y \\ -ay^l & -y & y + a^2\left(y^l + y^m\right) \end{bmatrix}}_{Y_1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

- $Y_1$ : complex symmetric
- Hence: admittance matrix with $\Pi$ circuit
- Unequal shunt elements

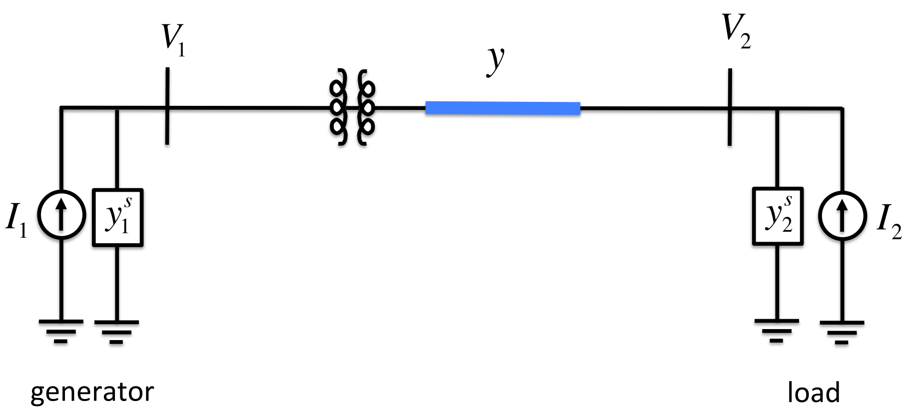
# Example

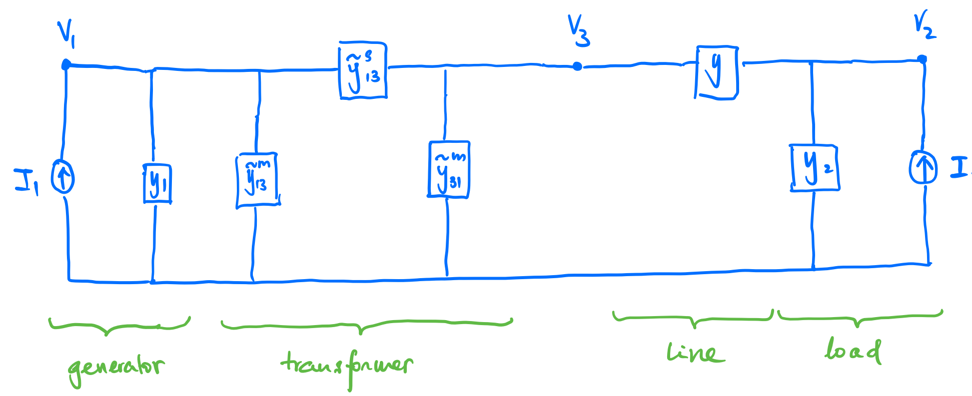
## Step 2: overall system




generator/load admittances

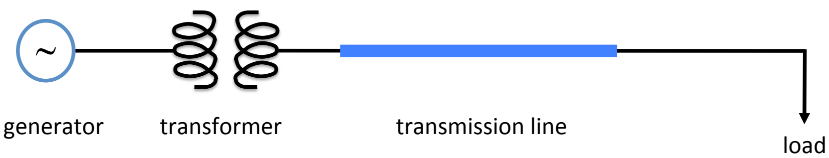
$$\begin{bmatrix} I_1\\ I_2\\ I_3\end{bmatrix} = \underbrace{\begin{bmatrix} y^l + \boxed{y_1} & 0 & -ay^l \\ 0 & y+\boxed{y_2} & -y \\ -ay^l & -y & y + a^2\left(y^l+y^m\right)\end{bmatrix}}_{Y} \begin{bmatrix} V_1\\ V_2\\ V_3\end{bmatrix}$$



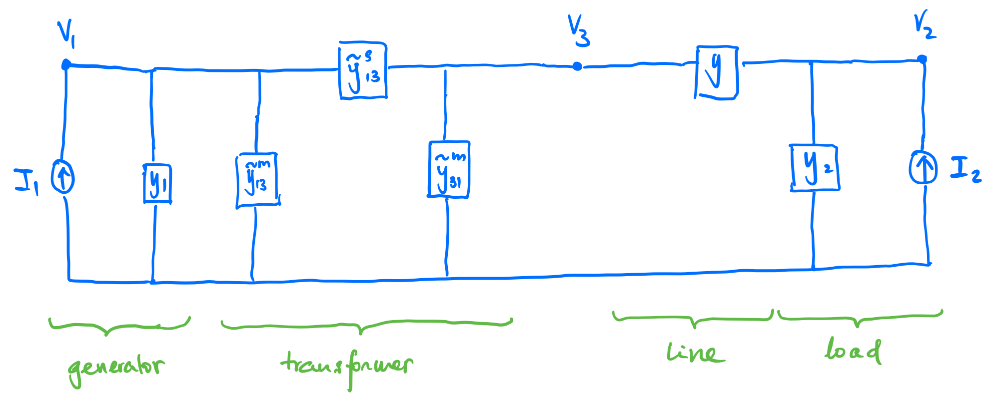
# Example

## Step 2: overall system




generator/load admittances

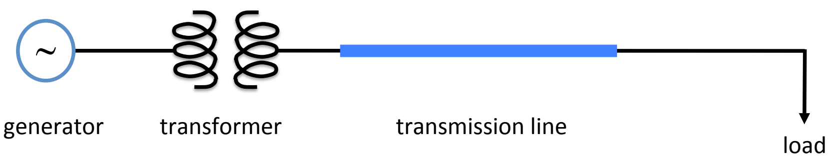
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} y^l + \boxed{y_1} & 0 & -ay^l \\ 0 & y + \boxed{y_2} & -y \\ -ay^l & -y & y + a^2\left(y^l + y^m\right) \end{bmatrix}}_{Y} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$




- Overall network model: **ideal** current sources connected by network
- Network: admittance matrix $Y$
- $Y$ includes admittances of non-ideal current sources

# Example

## Step 2: overall system

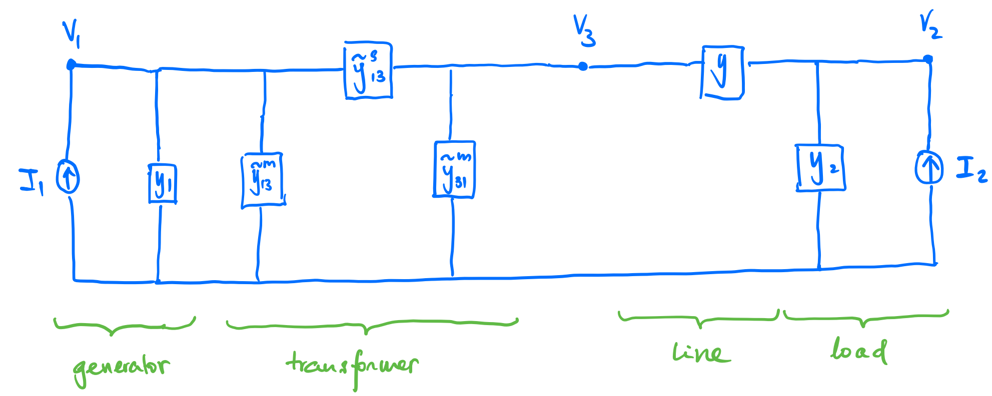



generator/load admittances

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} y^l + \boxed{y_1} & 0 & -ay^l \\ 0 & y + \boxed{y_2} & -y \\ -ay^l & -y & y + a^2\left(y^l + y^m\right) \end{bmatrix}}_{Y} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

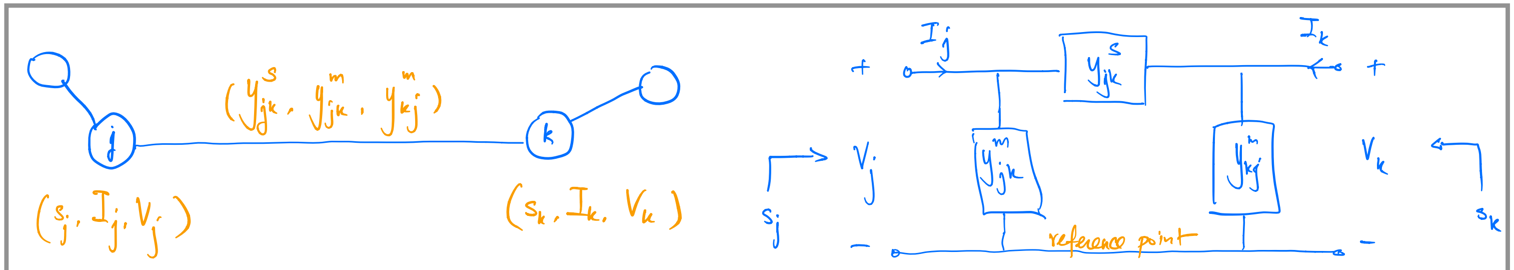
**Kron reduction** (see below)

- Internal bus has zero injection $I_3 = 0$
- Can eliminate $(V_3, I_3)$
- External behavior: relation between $(I_1, I_2)$ and $(V_1, V_2)$



# General network model

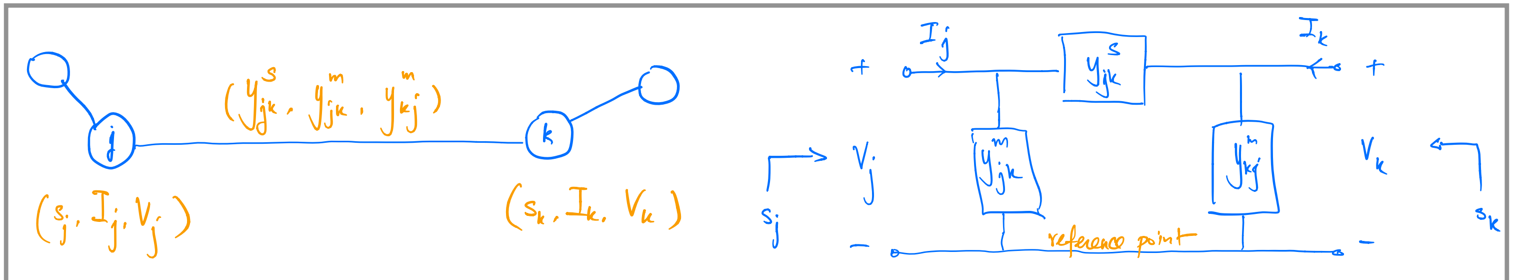
1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1, \ldots, N\}$ : buses/nodes/terminals
   - $E \subseteq \overline{N} \times \overline{N}$ : lines/branches/links/edges
2. Each line $(j, k)$ is parameterized by $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$
   - $y^s_{jk}$ : series admittance
   - $y^m_{jk}$, $y^m_{kj}$ : shunt admittances, generally different




(a) Graph representation (b) Π equivalent circuit

# General network model

## Branch currents



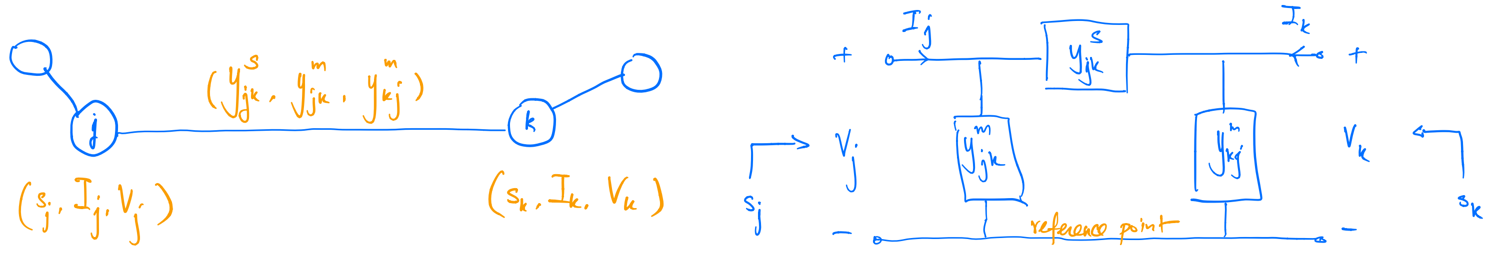

(a) Graph representation (b) Π equivalent circuit

Sending-end currents

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j, \qquad I_{kj} = y^s_{jk}(V_k - V_j) + y^m_{kj} V_k,$$

# $VI$ relation

## Nodal current balance

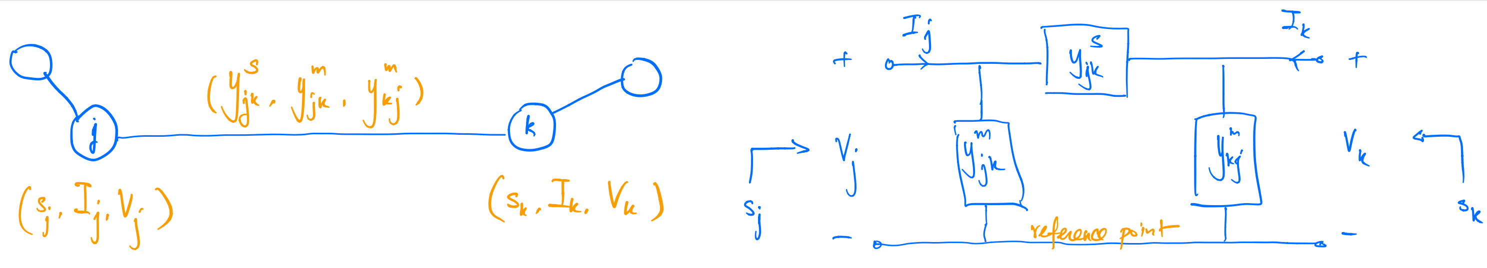



(a) Graph representation (b) Π equivalent circuit

$$I_j = \sum_{k:j\sim k} I_{jk}$$

# $VI$ relation

## Nodal current balance




(a) Graph representation (b) Π equivalent circuit

$$I_j = \sum_{k:j\sim k} I_{jk} = \left( \sum_{k:j\sim k} y^s_{jk} + y^m_{jj} \right) V_j - \sum_{k:j\sim k} y^s_{jk} V_k$$

total shunt admittance: $y^m_{jj} := \sum_{k:j\sim k} y^m_{jk}$

# $VI$ relation

## Admittance matrix $Y$

$$I_j = \sum_{k:j\sim k} I_{jk} \ = \ \left( \sum_{k:j\sim k} y^s_{jk} \ + \ y^m_{jj} \right) V_j \ - \ \sum_{k:j\sim k} y^s_{jk} V_k$$

In vector form:

$$I \ = \ Y V \text{ where } Y_{jk} = \begin{cases} -y^s_{jk}, & j \sim k \quad (j \neq k) \\ \sum_{l:j\sim l} y^s_{jl} \ + \ y^m_{jj}, & j = k \\ 0 & \text{otherwise} \end{cases}$$

# $VI$ relation

## Admittance matrix $Y$

$Y$ can be written down by inspection of network graph

- Off-diagonal entry: $-$ series admittance
- Diagonal entry: $\sum$series admittances $+$ total shunt admittance

In vector form:

$$I \;=\; Y\,V \;\; \text{where} \;\; Y_{jk} = \begin{cases} -y^s_{jk}, & j \sim k \quad (j \neq k) \\ \displaystyle\sum_{l:j\sim l} y^s_{jl} \;+\; y^m_{jj}, & j = k \\ 0 & \text{otherwise} \end{cases}$$

# $VI$ relation
## Admittance matrix $Y$

A matrix $Y$ is an admittance matrix iff it is complex symmetric

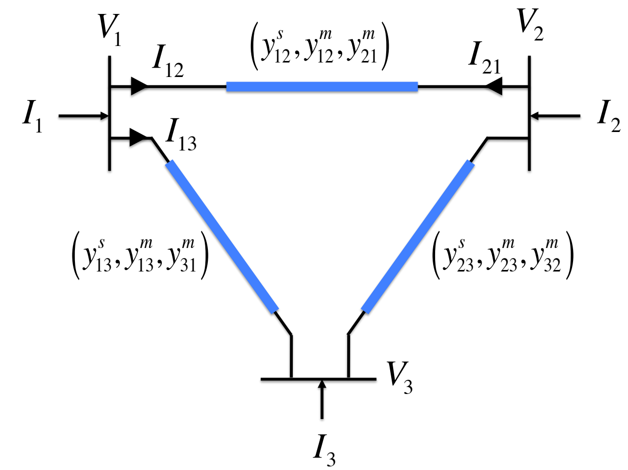
- Can be interpreted as a $\Pi$ circuit

In vector form:

$$I \;=\; YV \quad \text{where} \quad Y_{jk} = \begin{cases} -y^s_{jk}, & j \sim k \quad (j \neq k) \\ \sum_{l:j\sim l} y^s_{jl} \;+\; y^m_{jj}, & j = k \\ 0 & \text{otherwise} \end{cases}$$

# $VI$ relation

## Example




$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y^s_{12} + y^s_{13} + y^m_{11} & -y^s_{12} & -y^s_{13} \\ -y^s_{12} & y^s_{12} + y^s_{23} + y^m_{22} & -y^s_{23} \\ -y^s_{13} & -y^s_{23} & y^s_{13} + y^s_{23} + y^m_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

total shunt admittance: $y^m_{jj} \; := \; \sum_{k:j\sim k} y^m_{jk}$

# Admittance matrix $Y$

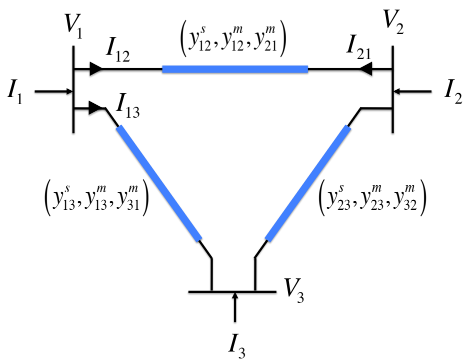
## In terms of incidence matrix $C$

bus-by-line incidence matrix

$$C_{jl} = \begin{cases} 1 & \text{if } l = j \rightarrow k \text{ for some bus } k \\ -1 & \text{if } l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

example:

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$



# Admittance matrix $Y$

## In terms of incidence matrix $C$

bus-by-line incidence matrix

$$C_{jl} = \begin{cases} 1 & \text{if } l = j \rightarrow k \text{ for some bus } k \\ -1 & \text{if } l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

$$Y = CY^sC^T + Y^m$$

where $Y^s := \text{diag}\left(y^s_{jk}\right), \; Y^m := \text{diag}\left(y^m_{jj}\right)$

$Y$ is a complex Laplacian matrix when $Y^m = 0$

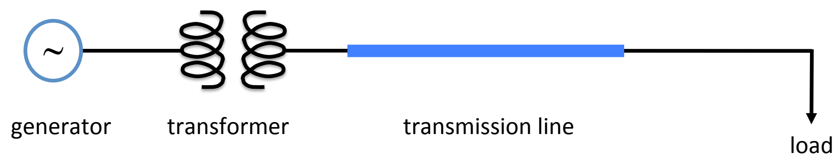
- See later for its properties

# Outline

1. Component models
2. Network model: $VI$ relation
   - Examples
   - $VI$ relation (admittance matrix $Y$)
   - Kron reduction
   - Invertibility of $Y$
3. Network model: $Vs$ relation
4. Computation methods

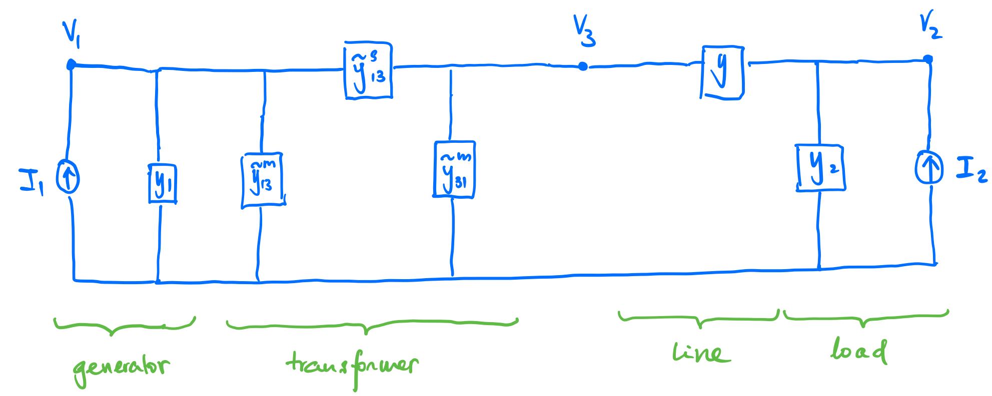
# Kron reduction

## Example




generator/load admittances

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} y^l + \boxed{y_1} & 0 & -ay^l \\ 0 & y + \boxed{y_2} & -y \\ -ay^l & -y & y + a^2\left(y^l + y^m\right) \end{bmatrix}}_{Y} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$




**Kron reduction**

- Internal bus has zero injection $I_3 = 0$
- External model relates $(I_1, I_2)$ and $(V_1, V_2)$
- Kron reduction: eliminate $(V_3, I_3)$

# Kron reduction

- $N_{\text{red}} \subseteq \overline{N}$ : buses of interest, e.g., terminal buses
- Want to relate current injections and voltages at buses in $N_{\text{red}}$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{Y} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$V_1$ ← $N_{\text{red}}$

$V_2$ ← $\overline{N} \backslash N_{\text{red}}$

- Eliminate $V_2 = -Y_{22}^{-1} Y_{21} V_1 + Y_{22}^{-1} I_2$
- Obtain: $\left( Y_{11} - Y_{12} Y_{22}^{-1} Y_{21} \right) V_1 \; = \; I_1 \; - \; Y_{12} Y_{22}^{-1} I_2$

Schur complement

# Kron reduction

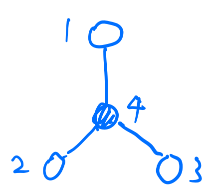
If internal injections $I_2 = 0$ :

$$\left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right) V_1 \;=\; I_1$$

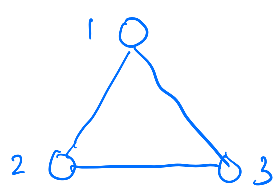
Schur complement

- Describes effective connectivity and line admittances of reduced network

Example:




original network




reduced network

# Invertibility of $Y$

## Zero shunt $Y^m = 0$

Admittance matrix $Y \;=\; CY^sC$ where $Y^s :=$ diag $\left(y^s_{jk}\right)$

When $Y$ is real, it is called a real Laplacian matrix

- $(N+1) \times (N+1)$ real symmetric matrix
- Row sum = column sum = 0
- rank$(Y) = N$, null$(Y) =$ span(1)
- Any principal submatrix is invertible

When $Y$ is a complex symmetric, but not Hermitian, these properties may not hold

# Invertibility of $Y$

## Zero shunt $Y^m = 0$

Admittance matrix $Y \;=\; CY^sC$ where $Y^s := \text{diag}\left(y^s_{jk}\right)$

**Theorem (Singular value decomposition)**

Suppose $Y^m = 0$. Then $Y = W\Sigma W^T$ where

- Unitary $W$ : columns are orthonormal eigenvectors of $Y\overline{Y}$ ($\overline{Y}$ : elementwise complex conjugate of $Y$)
- $\Sigma := \text{diag}\left(\sigma_j\right) : 0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_N$ are nonnegative roots of eigenvalues of $Y\overline{Y}$

Pseudo-inverse $Y^\dagger \;:=\; \overline{W}\Sigma^\dagger W^H$

- $\overline{W},\ W^H$ : elementwise complex conjugate and Hermitian transpose respectively of $W$
- $\Sigma^\dagger := \text{diag}\left(1/\sigma_j, j = 1,\ldots,N\right)$ where $1/\sigma_j := 0$ if $\sigma_j = 0$

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Admittance matrix $Y = CY^sC + Y^m$ where $Y^s := \text{diag}\left(y^s_{jk}\right)$, $Y^m := \text{diag}\left(y^m_{jj}\right)$

If $Y^s, Y^m$ are real symmetric such that

- $Y^s_{jk}, Y^m_{jj}$ are all of the same sign (e.g. for DC power flow model)

then

- $Y$ is strictly diagonally dominant: $|Y_{jj}| = \left| \sum_{k:j\sim k} y^s_{jk} + y^m_{jj} \right| > \sum_{k:j\sim k} |y^s_{jk}|, \ \forall j$
- $Y$ is therefore invertible and positive definite

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Sufficient (not necessary) condition for $Y^{-1}$ to exist is

$\alpha^H Y \alpha \neq 0$ for all $\alpha \in C^{N+1}$

Proof:

If $Y$ is not invertible then it has an eigenvector $\alpha$ with zero eigenvalue.

Hence $\alpha^H Y \alpha = 0$

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Sufficient (not necessary) condition for $Y^{-1}$ to exist is

$\alpha^H Y \alpha \neq 0$ for all $\alpha \in C^{N+1}$

$$\alpha^H Y \alpha = \sum_j \left( \left( \sum_{k:k\sim j} y_{jk}^s + y_{jj}^m \right) |\alpha_j|^2 - \sum_{k:k\sim j} y_{jk}^s \alpha_j^* \alpha_k \right)$$

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Sufficient (not necessary) condition for $Y^{-1}$ to exist is

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$$\alpha^H Y \alpha = \sum_j \left( \left( \sum_{k:k\sim j} y^s_{jk} + y^m_{jj} \right) |\alpha_j|^2 - \sum_{k:k\sim j} y^s_{jk} \alpha_j^* \alpha_k \right)$$

$$= \sum_{(j,k)\in E} y^s_{jk} \left( |\alpha_j|^2 - \alpha_j^* \alpha_k - \alpha_j \alpha_k^* + |\alpha_k|^2 \right) + \sum_{j \in \overline{N}} y^m_{jj} |\alpha_j|^2$$

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Sufficient (not necessary) condition for $Y^{-1}$ to exist is

$\alpha^H Y \alpha \neq 0$ for all $\alpha \in C^{N+1}$

$$\alpha^H Y \alpha = \sum_j \left( \left( \sum_{k:k\sim j} y_{jk}^s + y_{jj}^m \right) |\alpha_j|^2 - \sum_{k:k\sim j} y_{jk}^s \alpha_j^* \alpha_k \right)$$

$$= \sum_{(j,k)\in E} y_{jk}^s \left( |\alpha_j|^2 - \alpha_j^* \alpha_k - \alpha_j \alpha_k^* + |\alpha_k|^2 \right) + \sum_{j\in\overline{N}} y_{jj}^m |\alpha_j|^2$$

$$= \sum_{(j,k)\in E} y_{jk}^s \left| \alpha_j - \alpha_k \right|^2 + \sum_{j\in\overline{N}} y_{jj}^m |\alpha_j|^2$$

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Write $y^s_{jk} \;=:\; g^s_{jk} + i b^s_{jk}, \;\; y^m_{jj} \;=:\; g^m_{jj} + i b^m_{jj}$

$$\alpha^H Y \alpha = \left( \sum_{(j,k)\in E} g^s_{jk} \left| \alpha_j - \alpha_k \right|^2 + \sum_{j \in \overline{N}} g^m_{jj} \, |\alpha_j|^2 \right) +$$

$$i \left( \sum_{(j,k)\in E} b^s_{jk} \left| \alpha_j - \alpha_k \right|^2 + \sum_{j \in \overline{N}} b^m_{jj} \, |\alpha_j|^2 \right)$$

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Therefore $Y$ is invertible if

1. At least one shunt admittance $y_{jj}^m \neq 0$. All nonzero $g_{jj}^m$ (or $b_{jj}^m$) have same sign
2. All nonzero $g_{jk}^s$ (or $b_{jk}^s$) have same sign
3. All $g_{jk}^s \neq 0$. All nonzero $g_{jj}^m$ have the same sign as $g_{jk}^s$

$$\alpha^H Y \alpha = \left( \sum_{(j,k)\in E} g_{jk}^s \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in\overline{N}} g_{jj}^m \, |\alpha_j|^2 \right) + i \left( \sum_{(j,k)\in E} b_{jk}^s \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in\overline{N}} b_{jj}^m \, |\alpha_j|^2 \right)$$
$$\neq 0$$

If $(j, k)$ models a transmission line, then these sufficient conditions are satisfied

# Invertibility of $Y$

## Nonzero shunt $Y^m \neq 0$

Therefore $Y$ is invertible if

1. At least one shunt admittance $y_{jj}^m \neq 0$. All nonzero $g_{jj}^m$ (or $b_{jj}^m$) have same sign
2. All nonzero $g_{jk}^s$ (or $b_{jk}^s$) have same sign
3. All $g_{jk}^s \neq 0$. All nonzero $g_{jj}^m$ have the same sign as $g_{jk}^s$

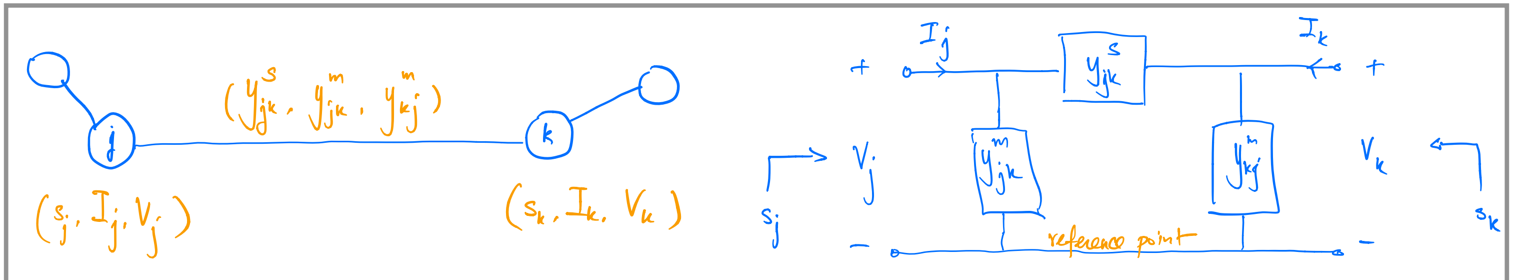
Similar argument leads to sufficient conditions on invertibility of $Y_{22}$ for Kron reduction

# Outline

1. Component models
2. Network model: $VI$ relation
3. Network model: $Vs$ relation
   - Complex form
   - Polar form
   - Cartesian form
   - Types of buses
4. Computation methods

# General network

## Branch currents



(a) Graph representation (b) Π equivalent circuit

Sending-end currents

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}\, V_j, \qquad I_{kj} \;=\; y^s_{jk}(V_k - V_j) \;+\; y^m_{kj}\, V_k,$$

# Power flow models

## Complex form

Using $S_{jk} := V_j I_{jk}^H$ :

$$S_{jk} = \left(y_{jk}^s\right)^H \left( |V_j|^2 - V_j V_k^H \right) + \left(y_{jk}^m\right)^H |V_j|^2$$

$$S_{kj} = \left(y_{jk}^s\right)^H \left( |V_k|^2 - V_k V_j^H \right) + \left(y_{kj}^m\right)^H |V_k|^2$$

# Power flow models

## Complex form

Bus injection model $s_j = \sum_{k:j\sim k} S_{jk}$ :

$$s_j = \sum_{k:j\sim k} \left(y_{jk}^s\right)^H \left(\,|V_j|^2 - V_j V_k^H\right) \;+\; \left(y_{jj}^m\right)^H |V_j|^2$$

In terms of admittance matrix $Y$

$$s_j = \sum_{k=1}^{N+1} Y_{jk}^H V_j V_k^H$$

$N+1$ complex equations in $2(N+1)$ complex variables $\left(s_j, V_j, j \in \overline{N}\right)$

# Power flow models

## Polar form

Write $s_j \;=:\; p_j + iq_j$ and $V_j =: |V_j| e^{i\theta_j}$ with $y^s_{jk} =: g^s_{jk} + ib^s_{jk}$, $y^m_{jk} =: g^m_{jk} + ib^m_{jk}$ :

$$p_j = \left(\sum_{k=0}^{N} g_{jk}\right) |V_j|^2 \;-\; \sum_{k\neq j} |V_j|\,|V_k| \left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right)$$

$$q_j = -\left(\sum_{k=0}^{N} b_{jk}\right) |V_j|^2 \;-\; \sum_{k\neq j} |V_j|\,|V_k| \left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right)$$

where $g_{jk} := \begin{cases} g^m_{jj} & \text{if } j = k \\ g^s_{jk} & \text{if } j \neq k,\ (j,k) \in E \\ 0 & \text{if } j \neq k,\ (j,k) \notin E \end{cases}$ $\qquad b_{jk} := \begin{cases} b^m_{jj} & \text{if } j = k \\ b^s_{jk} & \text{if } j \neq k,\ (j,k) \in E \\ 0 & \text{if } j \neq k,\ (j,k) \notin E \end{cases}$

# Power flow models

## Polar form

Write $s_j \;=:\; p_j + iq_j$ and $V_j =: |V_j|\, e^{i\phi}$ with $y^s_{jk} =: g^s_{jk} + ib^s_{jk}$, $y^m_{jk} =: g^m_{jk} + ib^m_{jk}$ :

$$p_j = \left(\sum_{k=0}^{N} g_{jk}\right)|V_j|^2 \;-\; \sum_{k\neq j} |V_j|\,|V_k| \left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right)$$

$$q_j = -\left(\sum_{k=0}^{N} b_{jk}\right)|V_j|^2 \;-\; \sum_{k\neq j} |V_j|\,|V_k| \left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right)$$

$2(N+1)$ real equations in $4(N+1)$ real variables $\left(p_j, q_j,\ |V_j|\,, \theta_j,\ j \in \overline{N}\right)$

# Power flow models

## Cartesian form

Write $s_j \;=:\; p_j + iq_j$ and $V_j =: e_j + if_j$ :

$$p_j = \left(\sum_k g_{jk}\right)\left(e_j^2 + f_j^2\right) - \sum_{k\neq j}\left(g_{jk}(e_je_k + f_jf_k) + b_{jk}(f_je_k - e_jf_k)\right)$$

$$q_j = -\left(\sum_k b_{jk}\right)\left(e_j^2 + f_j^2\right) - \sum_{k\neq j}\left(g_{jk}(f_je_k - e_jf_k) - b_{jk}(e_je_k + f_jf_k)\right)$$

$2(N+1)$ real equations in $4(N+1)$ real variables $\left(p_j,\, q_j,\, e_j, f_j,\, j \in \overline{N}\right)$

# Power flow models

## Types of buses

Power flow equations specify $2(N+1)$ real equations in $4(N+1)$ real variables

- Power flow (load flow) problem: given $2(N+1)$ values, determine remaining vars

Types of buses

- $PV$ buses : $\left(p_j, |V_j|\right)$ specified, determine $\left(q_j, \theta_j\right)$, e.g. generator
- $PQ$ buses : $\left(p_j, q_j\right)$ specified, determine $V_j$, e.g. load
- Slack bus $0$ : $V_0 := 1\angle 0^\circ$ pu specified, determine $\left(p_j, p_j\right)$

# Outline

1. Component models
2. Network model: $VI$ relation
3. Network model: $Vs$ relation
4. Computation methods
   - Gauss-Seidel algorithm
   - Newton-Raphson algorithm
   - Fast decoupled algorithm

# Computation methods

## Gauss-Seidel algorithm

*Case 1: given* $V_0$ *and* $\left(s_1, \ldots, s_N\right)$*, determine* $s_0$ *and* $(V_1, \ldots, V_N)$

Power flow equations

$$s_0 = \sum_k Y_{0k}^H V_0 V_k^H$$

$$s_j = \sum_k Y_{jk}^H V_j V_k^H, \qquad j \in N$$

- First compute $(V_1, \ldots, V_N)$
- Then compute $s_0$

# Computation methods

## Gauss-Seidel algorithm

*Case 1: given* $V_0$ *and* $\left(s_1, \ldots, s_N\right)$, *determine* $s_0$ *and* $(V_1, \ldots, V_N)$

Rearrange 2nd equation:

$$\frac{s_j^H}{V_j^H} = Y_{jj}V_j + \sum_{\substack{k=0 \\ k \neq j}}^{N} Y_{jk}V_k, \qquad j \in N$$

$$V_j = \frac{1}{Y_{jj}}\left(\frac{s_j^H}{V_j^H} - \sum_{\substack{k=0 \\ k \neq j}}^{N} Y_{jk}V_k\right) =: f_j\left(V_1, \ldots, V_N\right), \qquad j \in N$$

# Computation methods

## Gauss-Seidel algorithm

*Case 1: given* $V_0$ *and* $\left(s_1, \ldots, s_N\right)$, *determine* $s_0$ *and* $(V_1, \ldots, V_N)$

2nd power flow equation:

$$V = f(V)$$

where $V := \left(V_j, j \in N\right), \; f := \left(f_j, j \in N\right)$

Gauss algorithm is the fixed point iteration

$$V(t+1) = f(V(t))$$

# Computation methods

## Gauss-Seidel algorithm

*Case 1: given* $V_0$ *and* $\left(s_1, \ldots, s_N\right)$*, determine* $s_0$ *and* $(V_1, \ldots, V_N)$

Gauss algorithm:

$$
\begin{aligned}
V_1(t+1) &= f_1\left(V_1(t), \ldots, V_N(t)\right) \\
V_2(t+1) &= f_2\left(V_1(t), \ldots, V_N(t)\right) \\
&\vdots \\
V_N(t+1) &= f_N\left(V_1(t), \ldots, V_{N-1}(t), V_N(t)\right)
\end{aligned}
$$

# Computation methods

## Gauss-Seidel algorithm

*Case 1: given* $V_0$ *and* $\left(s_1, \ldots, s_N\right)$, *determine* $s_0$ *and* $(V_1, \ldots, V_N)$

Gauss-Seidel algorithm:

$$
\begin{aligned}
V_1(t+1) &= f_1\left(V_1(t), \ldots, V_N(t)\right) \\
V_2(t+1) &= f_2\left(V_1(t+1), \ldots, V_N(t)\right) \\
&\vdots \\
V_N(t+1) &= f_N\left(V_1(t+1), \ldots, V_{N-1}(t+1), V_N(t)\right)
\end{aligned}
$$

# Computation methods

## Gauss-Seidel algorithm

*Case 2: given* $(V_0, \ldots, V_m)$ *and* $(s_{m+1}, \ldots, s_N)$*, determine* $\left(s_j, j \leq m\right)$ *and* $(V_j, j > m)$

Power flow equations

$$s_j = \sum_k Y_{jk}^H V_j V_k^H, \qquad j \leq m$$

$$s_j = \sum_k Y_{jk}^H V_j V_k^H, \qquad j > m$$

- First compute $\left(V_{m+1}, \ldots, V_N\right)$ from 2nd set of equations using the same algorithm
- Then compute $\left(s_j, j \leq m\right)$ from 1st set of equations

# Computation methods

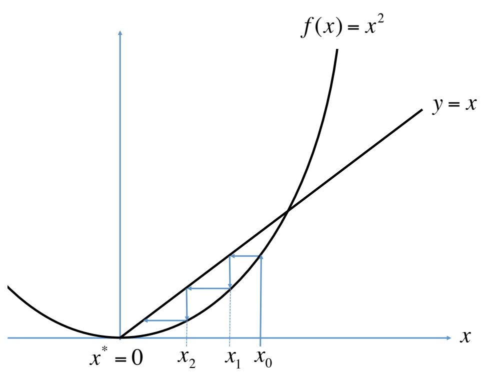
## Gauss-Seidel algorithm

If algorithm converges, the limit is a fixed point and a power flow solution

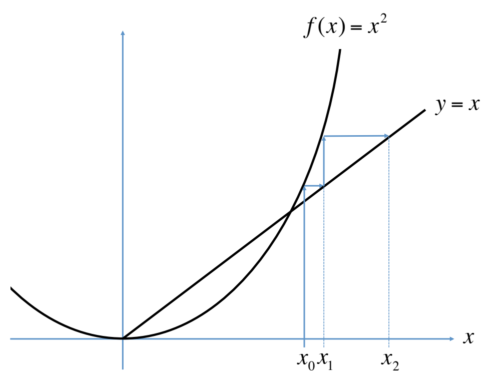
Algorithm converges linearly to unique fixed point if $f$ is a contraction mapping

- Contraction is sufficient, but not necessary, for convergence

In general, algorithm may or may not convergence depending on initial point




(a) Convergence




(b) Divergence

# Computational methods

## Newton-Raphson algorithm

To solve $$f(x) = 0$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, e.g. $\nabla F(x) = 0$ for unconstrained optimization

<u>Idea</u>:

- Linear approximation

$$\hat{f}(x(t+1)) = f(x(t)) + J(x(t))\, \Delta x(t)$$

- Choose $\Delta x(t)$ such that $\hat{f}(x(t+1)) = 0$, i.e., solve

$$J(x(t))\Delta x(t) = -f(x(t))$$

- Next iterate $x(t+1) := x(t) + \Delta x(t)$

$$J(x) := \frac{\partial f}{\partial x}(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}$$

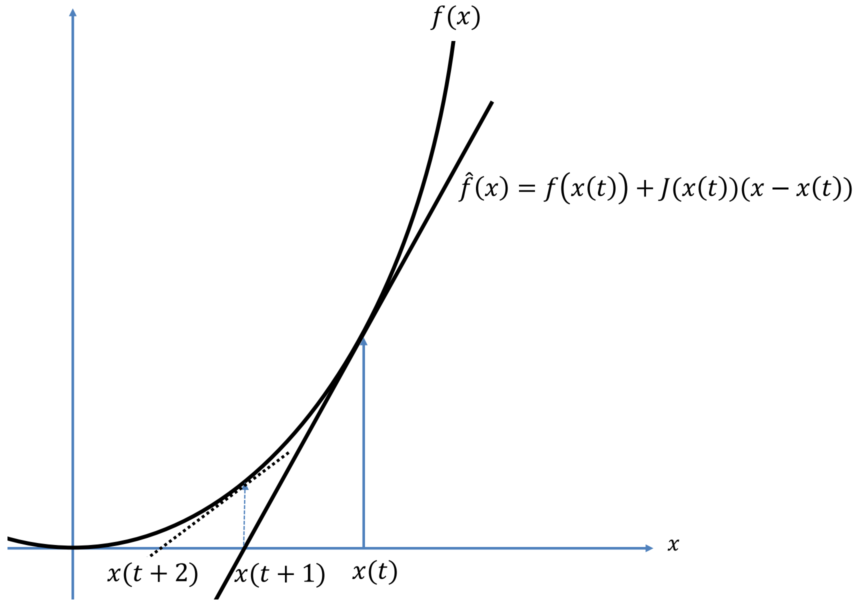
# Computational methods

## Newton-Raphson algorithm

To solve $$f(x) \;=\; 0$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, e.g. $\nabla F(x) = 0$ for unconstrained optimization

$$x(t+1) \;:=\; x(t) - \left(J(x(t))\right)^{-1} f(x(t))$$



# Computational methods

## Newton-Raphson algorithm

### Kantorovic Theorem

Consider $f : D \to \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ is an open convex set. Suppose

- $f$ is differentiable and $\nabla f$ is Lipschitz on $D$, i.e., $\|\nabla f(y) - \nabla f(x)\| \leq L\|y - x\|$
- $x_0 \in D$ and $\nabla f(x_0)$ is invertible

Let $\beta \geq \left\| \left( \nabla f(x_0) \right)^{-1} \right\|$, $\eta \geq \left\| \left( \nabla f(x_0) \right)^{-1} f(x_0) \right\|$ and

$$h := \beta\eta L, \quad r := \frac{1 - \sqrt{1 - 2h}}{h} \eta$$

# Computational methods

## Newton-Raphson algorithm

**Kantorovic Theorem**

Consider $f : D \to \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ is an open convex set. Suppose

- $f$ is differentiable and $\nabla f$ is Lipschitz on $D$, i.e., $\|\nabla f(y) - \nabla f(x)\| \leq L\|y - x\|$
- $x_0 \in D$ and $\nabla f(x_0)$ is invertible

If the closed ball $B_r(x_0) \subseteq D$ and $h \leq 1/2$, then Newton iteration

$$x(t+1) := x(t) - \left(\nabla f(x(t))\right)^{-1} f(x(t))$$

converges to a solution $x^* \in B_r\left(x_0\right)$ of $f(x) = 0$

Newton-Raphson converges if it starts close to a solution, often quadratically

# Computational methods

## Newton-Raphson algorithm

Apply to power flow equations in polar form:

$$p_j(\theta, |V|) = p_j, \qquad j \in N$$

$$q_j(\theta, |V|) = q_j, \qquad j \in N_{pq}$$

where

$$p_j(\theta, |V|) := \left(\sum_{k=0}^{N} g_{jk}\right)|V_j|^2 - \sum_{k \neq j} |V_j||V_k|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right)$$

$$q_j(\theta, |V|) := -\left(\sum_{k=0}^{N} b_{jk}\right)|V_j|^2 - \sum_{k \neq j} |V_j||V_k|\left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right)$$

# Computational methods

## Newton-Raphson algorithm

Define $f : \mathbb{R}^{N+N_{qp}} \rightarrow \mathbb{R}^{N+N_{qp}}$

$$f(\theta, |V|) \ := \ \begin{bmatrix} \Delta p(\theta, |V|) \\ \Delta q(\theta, |V|) \end{bmatrix} \ := \ \begin{bmatrix} p(\theta, |V|) - p \\ q(\theta, |V|) - q \end{bmatrix}$$

with

$$J(\theta, |V|) := \begin{bmatrix} \frac{\partial p}{\partial \theta} & \frac{\partial p}{\partial |V|} \\ \frac{\partial q}{\partial \theta} & \frac{\partial q}{\partial |V|} \end{bmatrix}$$

# Computational methods

## Newton-Raphson algorithm

1. Initialization: choose $\left(\theta(0), |V(0)|\right)$

2. Iterate until stopping criteria

   (a) Determine $\left(\Delta\theta(t), \Delta|V|(t)\right)$ from

$$J\left(\theta(t), |V|(t)\right) \begin{bmatrix} \Delta\theta(t) \\ \Delta|V|(t) \end{bmatrix} = - \begin{bmatrix} \Delta p(\theta(t), |V|(t)) \\ \Delta q(\theta(t), |V|(t)) \end{bmatrix}$$

   (b) Set

$$\begin{bmatrix} \theta(t+1) \\ |V|(t+1) \end{bmatrix} := \begin{bmatrix} \theta(t) \\ |V|(t) \end{bmatrix} + \begin{bmatrix} \Delta\theta(t) \\ \Delta|V|(t) \end{bmatrix}$$

# Computational methods

## Fast Decoupled algorithm

Key observation: the Jacobian is roughly block-diagonal

$$J(\theta, |V|) := \begin{bmatrix} \frac{\partial p}{\partial \theta} & \frac{\partial p}{\partial |V|} \\ \frac{\partial q}{\partial \theta} & \frac{\partial q}{\partial |V|} \end{bmatrix} \approx \begin{bmatrix} \frac{\partial p}{\partial \theta} & 0 \\ 0 & \frac{\partial q}{\partial |V|} \end{bmatrix}$$

i.e., decoupling between $p$ and $|V|$, and between $q$ and $\theta$

# Computational methods

## Fast Decoupled algorithm

Key observation: the Jacobian is roughly block-diagonal

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i.e., decoupling between $p$ and $|V|$, and between $q$ and $\theta$

This simplifies the computation of $\Delta x(t)$

$$\frac{\partial p}{\partial \theta}(\theta(t), |V|(t)) \ \Delta\theta(t) = -\Delta p(\theta(t), |V|(t))$$

$$\frac{\partial q}{\partial |V|}(\theta(t), |V|(t)) \ \Delta|V|(t) = -\Delta q(\theta(t), |V|(t))$$

# Computational methods

## Fast Decoupled algorithm

Decoupling assumption: $g_{jk} = 0, \sin\theta_{jk} = 0$

$$\frac{\partial p_j}{\partial |V_k|} = \begin{cases} -|V_j|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k \\ \dfrac{p_j(\theta, |V|)}{|V_j|} + \left(\sum_i g_{ji}\right)|V_j|, & j = k \end{cases}$$

$$g_{jk} = 0, \ \sin\theta_{jk} = 0, \ p_j(\theta, |V|) = 0 \quad \Rightarrow \quad \frac{\partial p}{\partial |V|} = 0$$

# Computational methods

## Fast Decoupled algorithm

Decoupling assumption: $g_{jk} = 0, \sin\theta_{jk} = 0$

$$\frac{\partial q_j}{\partial \theta_k} = \begin{cases} |V_j||V_k|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k \\ p_j(\theta, |V|) - \left(\sum_i g_{ji}\right)|V_j|^2, & j = k \end{cases}$$

$$g_{jk} = 0, \ \sin\theta_{jk} = 0, \ p_j(\theta, |V|) = 0 \quad \Rightarrow \quad \frac{\partial q}{\partial \theta} = 0$$

# Summary

1. Component models
   - Single-phase devices, line, transformer
2. Network models
   - $VI$ relation (admittance matrix $Y$), $Vs$ relation (power flow models)
3. Computation methods
   - Gauss-Seidel algorithm, Newton-Raphson algorithm, Fast decoupled algorithm